CHAPTER-35

THE IMPACT OF POLLUTED ENVIRONMENT ON POPULATION DYNAMICS OF PREDATOR AND PREY: INFERENCE OF LOTKA-VOLTERRA MODEL

[1]Dr. UDAY RAJ SINGH, [2]SATENDRA KUMAR SUMAN, [3]Dr. USHA SINGH

[1]*Associate Professor, Department of Mathematics, C.L. Jain (PG) College, Firozabad (affiliated to Dr. B.R.A. University), Agra (U.P.), India*

[2]*Research Scholar, C.L. Jain (PG) College, Firozabad (affiliated to Dr. B.R.A. University), Agra (U.P.), India*

[3]*Associate Professor, Department of Mathematics, C.L. Jain (PG) College, Firozabad (affiliated to Dr. B.R.A. University), Agra (U.P.), India*

Chapter ID: NSP/ICAAR-2023/A-35

ABSTRACT

It can be difficult to address in practice the dynamics that drive growth, development, survival, and change within an ecological system when a predator and prey species coexist. Mathematical models have helped ecologists better comprehend the system's dynamics. For this purpose, the Lotka-Volterra model, a mathematical framework developed by Alfred J. Lotka and Vito Volterra, is employed. Adaptations were made to the prevailing ecological conditions. The Lotka-Volterra equations are revised as a result of these presumptions. The improved model's equilibrium and stability features are proven. The outcomes were modelled in MATLAB. In order to create the mathematical model, nonlinear ordinary differential equations are used. Biological and chemical toxins are assumed to stunt the development of prey species in mathematical models. We also talk about the model's equilibrium points. Positive definite functions and the Jacobian matrix are taken into account to evaluate the mathematical model. Computational models corroborate the analytical assumptions.

Keyword: *Equilibrium points, local and global stability, variational matrix, Lotka volterra model.*

1. INTRODUCTION

Modern industrial, agricultural, and other human activities have contributed to one of the most pressing socio-ecological challenges in the world: environmental pollution. When a toxic substance is present in the environment, it poses a serious risk to the wellbeing of any organisms in it. This encourages researchers to examine population resilience in contaminated environments and determine the critical levels of pollution at which populations begin to decline.

The ecology is negatively impacted by the natural environment. Fact is, industries are a major contributor to the widespread nature of various contaminations. Water, smoke, and garbage are just some of the byproducts of industry. Which are mostly to blame for the widespread dispersal of chemicals with extremely negative effects on our ecosystem, and which have seriously polluted the atmosphere and waterways as a result of these releases. Toxins and contaminants like cadmium, zinc, mercury, etc., are flourishing in our natural lakes and oceans, endangering a wide variety of animal and plant life. Biologists and mathematicians are very curious about how poisons affect populations in ecological food chains. Every ecosystem has a selvage where prey and predator interactions can be seen, providing a rich opportunity to study these connections from several angles. The natural behavior and interaction between biological population systems have been extrapolated using a wide variety of prey predator functional settings. Many different areas of prey-predator interactions have been studied by many different biologists, mathematicians, and academics over the course of many decades. Because a shift in the behavioral way interacts between prey and predator can change the whole gesture of the biological species system, understanding that the predator eats the prey is a crucial part of dealing with the interactions in prey-predator systems.

We analyze the mathematical underpinnings of a model provided by Chauhan and Mishra (2012) to investigate the effects of pollution and virus-induced illness on populations of animals within a single species. It was found that susceptible populations do not simply vanish when struck by virus, but that they can go extinct due to environmental pollution. Haung et al. (2015) developed a toxin-dependent predator-prey model to examine the effects of environmental toxins on the dynamics of food webs. The model integrated direct and indirect toxic effects on two trophic levels. They investigated the impact that changes in environmental toxicity have on the stability of the classic predator-prey dynamic. There was evidence that high levels of toxins had a deleterious effect on both species, possibly leading to their extinction. Mishra et al. (2015) explored a model for a three-species system with a time lag, which included two competing species and a predator species that is partially related with an alternative prey. They show that the abundance of all species fluctuates in response to shifts in the availability of a common food source. Misra and Annavarapu (2016) investigated the food web in polluted environments, and their results show that the predator rate of the intermediate predator is a bifurcation parameter, with Hopf-bifurcation occurring at some critical value of this parameter. Mishra and Annavarapu (2016) addressed the broader systemic effects of a toxicant on a three-species food web. The models assume that the intermediate predator's capacity to prey upon its victim is reduced in the presence of a top predator. Stability analysis of the models established the thresholds below which populations under toxicant stress cease to exist. Wang et al. (2017) formulated and investigated an eco-evolutionary resource-consumer-predator trophic cascade model that incorporates rapid evolution. In addition, the authors were interested in how a density-mediated indirect effect influences the evolution of populations and traits. Hadziabdic et al. (2017) laid out the conditions under which extinction of one predator is inevitable and coexistence of others is possible. They considered what would occur when one integral curve was placed next to another integral curve. A consistent rate of harvesting of diseased predators by prey and of susceptible predators is accounted for in Themairi and Alqudah's (2020) improved Holling-type II predator-prey model. Researchers established the existence of both uniform boundedness and positive biological equilibrium. Wang et al. (2021) examined the impact of pollution on the body size of prey in a predator-prey evolutionary model with a continuous phenotypic feature under conditions of intermittent pollution discharge. Fast evolution, they showed, can enhance prey density and protect populations from extinction, but that it loses its evolutionary advantages as pollution rises. Rat and cat population dynamics variables were added to the Lotka-Volterra Predator-Prey Model by Ogethakpo and Ojobor (2021). Predator-prey interactions were found to be affected by the introduction of artificial noise in their simulations. Gao and Yao (2022) developed a stochastic model of a predator-prey relationship based on a variant of the Leslie-Gower Holling type II.

2. MATHEMATICAL MODELING

To better understand how organisms might thrive in a contaminated environment, a nonlinear mathematical model was developed. The state variables of the following nonlinear mathematical model are the densities of prey $x_1(t)$, predators $x_2(t)$, organism toxicant $u(t)$ and environmental toxicant $v(t)$.

It is possible to express the Lotka-Volterra equations in a straightforward manner by writing them as a system of first-order non-linear ordinary differential equations (ODEs). As a result of the differential character of the equations, the solutions are deterministic (there is no element of chance involved, and maintaining the same beginning conditions will always generate the same result), and time is continuous (the generations of predators and prey are always overlapping one another). When developing the Lotka-Volterra equations, several assumptions were made, as was the case with the development of most other mathematical models. These presumptions include the following:

(i) There is an abundant supply of food available for the population of predators.
(ii) The size of the population of the prey has a direct bearing on the total amount of food that is made available to the prey.
(iii) The size of the population has a direct bearing on the rate at which it grows and shrinks.
(iv) The environment is considered to be stable, and the possibility of genetic adaptation is not discounted out of hand.
(v) Predators will never stop devouring their prey.

Following the establishment of such presumptions, we extend the Lotka–Volterra model as follows:

$$\frac{dx_1}{dt} = c_1x_1 - c_1{x_1}^2 - c_2x_1x_2 - r_1vx_1 - d_1x_1 \quad (1)$$

$$\frac{dx_2}{dt} = c_3x_1x_2 - c_4x_2 - r_2vx_2 - d_2x_2 \quad (2)$$

$$\frac{du}{dt} = -hu + q \quad (3)$$

$$\frac{dv}{dt} = a_1u + \frac{dn\phi}{a_1} - (l_1 + l_2)v \quad (4)$$

Table 1: Model parameters having biological significance

S.No.	Parameters	Connotation in biology
1.	h	Pollution's effect on the rate at which prey populations are declining
2.	q	Rate of pollutant release into the environment from outside sources remains unchanged
3.	l_1, l_2	The relative rates of pollutant net intake and excretion in living organisms
4.	a_1	Concentration of environmental pollutants in a given organism
5.	d	Concentration of contaminants in organisms per unit of mass consumed
6.	n	Pollutant load in the system
7.	ϕ	The Typical feeding rate of an organism
8.	d_1	Prey mortality due to natural causes
9.	d_2	The predator mortality rate
10.	h	Pollution's effect on the rate at which prey populations are declining

3. EXISTENCE OF EQUILIBRIUM POINT

$$c_1x_1 - c_1{x_1}^2 - c_2x_1x_2 - r_1vx_1 - d_1x_1 = 0 \quad (5)$$

$$c_3x_1x_2 - c_4x_2 - r_2vx_2 - d_2x_2 = 0 \quad (6)$$
$$-hu + q = 0 \quad (7)$$
$$a_1u + \frac{dn\phi}{a_1} - (l_1 + l_2)v = 0 \quad (8)$$

1) **Existence of $E_1(x_1, 0, u, v)$:** In the absence of predator i.e. $x_2 = 0$

$$x_1 = \frac{ha_1(c_1-d_1)(l_1+l_2)-r_1(qa_1{}^2+hdn\phi)}{c_1a_1h(l_1+l_2)}$$
$$x_2 = 0$$
$$u = \frac{q}{h}$$
$$v = \frac{qa_1{}^2+hdn\phi}{ha_1(l_1+l_2)}$$

2) **(3.2) Existence of $E_2(x_1, x_2, u, v)$:**

$$x_1 = \frac{ha_1(c_4+d_2)(l_1+l_2)+r_2(qa_1{}^2+hdn\phi)}{ha_1c_3(l_1+l_2)}$$
$$x_2 = \frac{ha_1(l_1+l_2)[(c_1-d_1)c_3-(c_4+d_2)c_1]-(qa_1{}^2+hdn\phi)(r_2c_1+r_1c_3)}{ha_1c_2c_3(l_1+l_2)}$$
$$u = \frac{q}{h}$$
$$v = \frac{qa_1{}^2+hdn\phi}{ha_1(l_1+l_2)}$$

4. ANALYSIS OF LOCAL STABILITY

We have the variation matrix for the system (1)-(4) as

$$J(E) = \begin{bmatrix} c_1 - 2c_1x_1 - c_2x_2 - r_1v - d_1 & -c_2x_1 & 0 & -r_1x_1 \\ c_3x_2 & c_3x_1 - c_4 - r_2v - d_2 & 0 & -r_2x_2 \\ 0 & 0 & -h & 0 \\ 0 & 0 & a_1 & -(l_1 + l_2) \end{bmatrix}$$

1) **The variation matrix for $E_1(x_1, 0, u, v)$**

$$J(E_1) = \begin{bmatrix} c_1 - 2c_1x_1 - r_1v - d_1 & -c_2x_1 & 0 & -r_1x_1 \\ 0 & c_3x_1 - c_4 - r_2v - d_2 & 0 & 0 \\ 0 & 0 & -h & 0 \\ 0 & 0 & a_1 & -(l_1 + l_2) \end{bmatrix}$$

The Eigen values of the variational matrix are

$$\lambda_1 = \frac{ha_1(l_1+l_2)[c_1c_3-(c_1c_4+c_3d_1+c_1d_2)]-(qa_1{}^2+hdn\phi)(c_3r_1+c_1r_2)}{c_1a_1h(l_1+l_2)}$$
$$\lambda_2 = \frac{r_1(qa_1{}^2+hdn\phi)-ha_1(c_1-d_1)(l_1+l_2)}{ha_1(l_1+l_2)}$$
$$\lambda_3 = -h$$
$$\lambda_3 = -(l_1 + l_2)$$

All the Eigen values of variation matrix are negative if

$$d_1 + \frac{r_1(qa_1{}^2+hdn\phi)}{ha_1(l_1+l_2)} < c_1 < \frac{(qa_1{}^2+hdn\phi)(c_3r_1+c_1r_2)}{c_3ha_1(l_1+l_2)} + (c_1c_4 + c_3d_1 + c_1d_2)$$

If the aforementioned requirements are met, then E_1 is deemed to be locally asymptotically stable, per the criteria of Routh and Hurwitz.

2) **The variation matrix for $E_2(x_1, x_2, u, v)$**

$$J(E_2) = \begin{bmatrix} c_1 - 2c_1x_1 - c_2x_2 - r_1v - d_1 & -c_2x_1 & 0 & -r_1x_1 \\ c_3x_2 & c_3x_1 - c_4 - r_2v - d_2 & 0 & -r_2x_2 \\ 0 & 0 & -h & 0 \\ 0 & 0 & a_1 & -(l_1 + l_2) \end{bmatrix}$$

The Eigen values of the variational matrix are

$\lambda_1 = \frac{i}{2}\sqrt{4bw}$

$\lambda_2 = \frac{i}{2}\sqrt{4bw}$

$\lambda_3 = -h$

$\lambda_4 = -(l_1 + l_2)$

Where

$$b = \frac{4c_2ha_1(c_4+d_2)(l_1+l_2)+r_2(qa_1{}^2+hdn\phi)}{ha_1c_3(l_1+l_2)}$$

$$w = \frac{ha_1(l_1+l_2)[c_1c_3-(c_4c_1+d_2c_1+d_1c_3)]-(qa_1{}^2+hdn\phi)(r_2c_1+r_1c_3)}{ha_1c_2(l_1+l_2)}$$

If these conditions hold, then E_2 is deemed to be locally asymptotically stable, under the criteria of Routh and Hurwitz.

$$\frac{c_1}{c_3} = \frac{(qa_1{}^2+hdn\phi)r_1}{ha_1(c_4+d_2)(l_1+l_2)}$$

And $c_1c_3 > \frac{(qa_1{}^2+hdn\phi)(r_2c_1+r_1c_3)}{ha_1(l_1+l_2)} + (c_4c_1 + d_2c_1 + d_1c_3)$

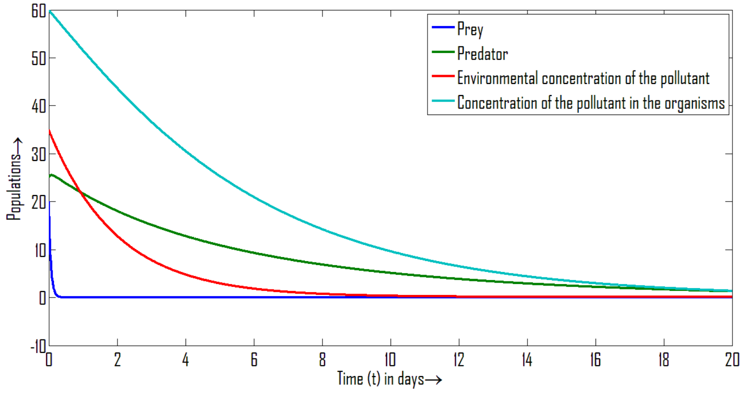
5. SIMULATION OF THE MODEL

We use numerical methods to probe the system's dynamics. The following parameters, all of which are hypothetical but biologically possible, are taken into account:

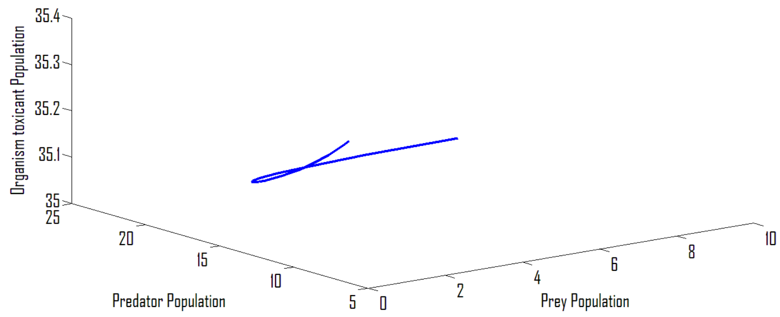
$d_1 = 0.7, d_2 = 2, r_1 = 0.2, r_2 = 0.1, l_1 = 0.1, l_2 = 0.3, d = 0.2, n = 0.5$
$, \phi = 0.3, c_1 = 2, c_2 = 0.2, c_3 = 3, c_4 = 2, a_1 = 2, q = 3, h = 2$

Graph (1) depicts how the model's various population densities change over time. The number of potential prey is low, and the amount of pollution in the organism is high. Graphs (2)–(4) depict three-dimensional maps of varying population densities.

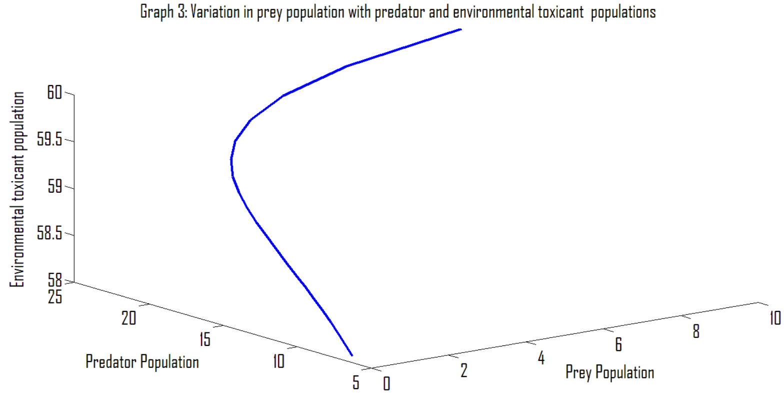
Graph 1: Population densities with time t



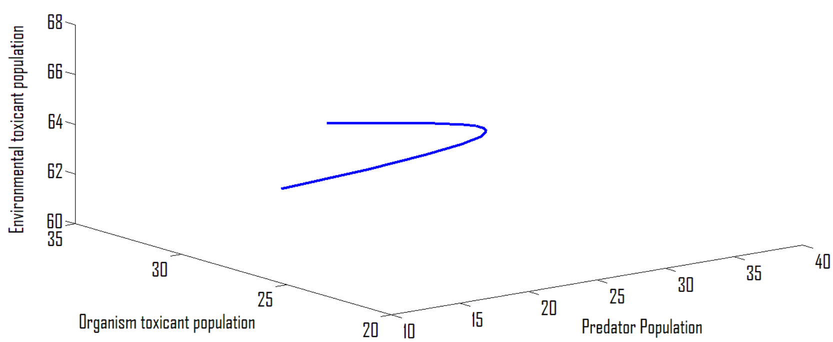
Graph 2: Variation in prey population with predator and organism toxicant populations



Graph 3: Variation in prey population with predator and environmental toxicant populations



Graph 4: Variation in Predator population with organism toxican and environmental toxicant populations



6. CONCLUDING REMARKS

This work proposes and investigates a prey-predator paradigm in a contaminated environment. Both populations are thought to be exposed to the pollution, but scientists know that only prey populations are at risk. The solutions' stability was then analyzed. At the same time as the number of sick prey decreases when the exogenous input rate of pollutant into the environment is reduced below a certain threshold, the number of healthy prey in the system rises. In contrast, the presence of the pollutant causes a decline in the number of predators, as is seen in the actual world. In order to learn which parameters govern the system's dynamic behavior, numerical simulations are performed. The selected criteria were hypothetical and consistent with biological reality. Pollutants may have beneficial effects on the survival of some species, at least according to the data that was examined.

REFERENCES

1. *Chauhan S., Mishra O.P. (2012): "Modeling and a analysis of a single species population with viral infection in polluted environment", Applied Mathematics, 3: 662-672.*
2. *Gao Y., Yao S. (2022): "Dynamical analysis of a modified Leslie–Gower Holling-type II predator-prey stochastic model in polluted environments with interspecific competition and impulsive toxicant input", Journal of Biological Dynamics, 16(1):840-858.*
3. *Hadziabdic V., Mehuljic M., Bekteseviv J. (2017): "Lotka-volterra model with two predators and their prey", TEM Journal, 6(1):132-136.*
4. *Huang Q., Wang H., Lewis M.A. (2015): "The impact of environmental toxins on predator–prey dynamics", Journal of Theoretical Biology: 378:12-30.*
5. *Mishra O.P., Annavarapu R.B. (2016): "A model for the dyanamical study of food chain system considering intereference of top predator in a polluted environment", Journal of Mathematical Modeling, 3 (2):189-218.*
6. *Mishra O.P., Annavarapu R.B. (2016): "A model for the dynamical study of food–chain system considering interference of top predator in a polluted environment", Journal of Mathematical Modeling, 3(2):189-218.*
7. *Mishra O.P., Sinha P., Sikarwar C.S. (2015): "Dynamical study of prey predator system with a commensal species competing with prey species: effect of time lag and alternative food source", Computational and Applied Mathematics, 34: 343-361.*
8. *Ogethakpo A.J., Ojobor S.A. (2021): " The lotka-volterra predator-prey model with disturbance", Nigerian Journal of Science and Environment, 19(2): 135-144.*
9. *Themairi A. A., Alqudah M.A. (2020): "Predator-prey model of Holling-type II with harvesting and predator in disease", Italian journal of pure and applied mathematics, 43:744-753.*
10. *Wang X., Fan M., Hao L.N. (2017): "Adaptive evolution of body size subject to indirect effect in trophic cascade system", Biosystems, 159:23–35.*
11. *Wang X., Song Le, Jingna L. (2021): "Study on evolution of a predator–prey model in a polluted environment", Nonlinear Analysis: Modelling and Control, 26(6):1052-1074.*