CHAPTER-33

RELEVANCE OF THE ETHPM TO THE NUMERICAL SOLUTION OF NONLINEAR SPACE-TIME FRACTIONAL EQUATION OF FOKKER-PLANCK

[1]DIWARI LAL, [2]KARM VIR

[1]Professor, Narain (P.G.) College, Shikohabad, Uttar Pradesh, Affiliated to Dr. Bhimrao Ambedkar University, Agra, India

[2]Research Scholar, Narain (P.G.) College, Shikohabad, Uttar Pradesh, Affiliated to Dr. Bhimrao Ambedkar University, Agra, India

Chapter ID: NSP/ICAAR-2023/A-33

ABSTRACT

In this study, we show how the conformable Elzaki transform Homotopy perturbation method (ETHPM) may be used to resolve the non-linear time-fractional equation of Fokker-planck. Four distinct varieties of non-linear time-fractional FPE are solved using the proposed method. Finally, a quick look at the numerical findings is taken at several different values of. The proposed strategy is both easy to implement and effective in practice.

Keywords: *FPE, ETHPM, non-linear time factional, Brownian motion*

INTRODUCTION

The Fokker–Planck equation is utilized not just in numerous subfields of physics and mathematics, but also in quantum optics, chemical physics, theoretical biology, and even circuit theory. In order to provide an explanation for Brownian motion, Fokker and Planck devised the Fokker-Planck equation. FPE is used to explain the spatial and temporal evolution of a probability distribution, and one obvious application of this concept is in the field of solute transport.

Using a computer simulation, Magdziarz et al. (2007) presented a stochastic representation of the fractional Fokker-Planck equation that represents anomalous diffusion. This representation was derived from the equation after it was ran through the computer.

Garg and Manohar (2011) used the generalized differential transform method (GDTM) to offer closed-form solutions to a number of linear and nonlinear space-time fractional Fokker-Planck equations. These solutions were found by using a variety of space-time fractional Fokker-Planck equations. (FPE). Jassim (2013) utilized a modified variational iteration approach in order to calculate the space-time fractional FPE. Using the homotopy perturbation approach, which Dubey et al. (2015) suggest by making use of the Sumudu transform, a variety of linear and nonlinear space-time fractional Fokker-Planck equations (FPEs) can be solved in closed form using the homotopy perturbation method. In the study that was published in 2015 by Firoozjaee et al., a straightforward approach for solving the Fokker-Planck equation in the presence of space- and time-fractional derivatives was presented. Prakash and Kaur (2017) proposed a hybrid computational methodology that is an imaginative reconciliation of q-homotopy analysis method (q-HAM) and Laplace transform in order to achieve the solution of time-fractional and space-time-fractional FPE. Hemeda and Eladdad (2018) proposed a new iterative approach and presented the integral iterative method as a means of solving linear and nonlinear Fokker-Planck equations as well as equations linked to these types of equations. The fractional Fokker-Planck equation was given as a system of nonlinear equations by Habenom and Suthar (2020). These authors used the shifted Chebyshev collocation method and the finite difference method to solve the problem. (FDM). Nuntadilok (2021) uses the homotopy perturbation approach in conjunction with the natural transform in order to solve a number of nonlinear space-time fractional Fokker-Planck equations. It uses a conformable version of the Elzaki transform. Iqbal et al. (2022) presented the homotopy perturbation method as a way for solving important classes of non-linear time-fractional partial differential equations. This method was developed in order to solve these problems. Lal and Vir (2022) proposed a new method that combines the Laplace transform with homotopy perturbation techniques as a solution to the difficulties of solving the non-linear Fokker-Planck equation. This was done in an effort to address the difficulty of the equation's solution.

PRINCIPLE BEHIND THE HOMOTOPY PERTURBATION APPROACH OF THE ELZAKI TRANSFORM (ETHPM)

Take the fractional nonlinear space-time Fokker-Planck equation as an example

$$\frac{\partial^\alpha U}{\partial t^\alpha} = \left[-\frac{\partial^\beta P(x,t,U)}{\partial x^\beta} + \frac{\partial^{2\beta} Q(x,t,U)}{\partial x^{2\beta}}\right] u(x,t), \alpha > 0, \beta < 1 \quad (1)$$

With the starting point in mind $U(x,0) = g(x)$ (2)

By subjecting (1) and (2) to an Elzaki transform with the same beginning conditions, we get

$$E[U(x,t)] = g(x) + s^\alpha E\left[-\frac{\partial^\beta P(x,t,U)}{\partial x^\beta} U(x,t) + \frac{\partial^{2\beta} Q(x,t,U)}{\partial x^{2\beta}} U(x,t)\right] \quad (3)$$

Using the Elzaki transform in reverse to solve the problem (3)

$$U(x,t) = g(x) + E^{-1}\left[s^\alpha E\left\{-\frac{\partial^\beta}{\partial x^\beta}\sum_{r=0}^{\infty} P_r + \frac{\partial^{2\beta}}{\partial x^{2\beta}}\sum_{r=0}^{\infty} Q_r\right\}\right] \quad (4)$$

Equation (4) can be solved with the homotopy perturbation approach.

$$\sum_{r=0}^{\infty} a^n U(x,t) = g(x) + a\, E^{-1}[s^\alpha E\{-\sum_{r=0}^{\infty} P_r H_r(x,t,U) + \sum_{r=0}^{\infty} Q_r h_r\,(x,t,U)\}] \quad (5)$$

where the nonlinear terms are represented by He's polynomials $H_r(x,t,U)$ and $h_r(x,t,U)$.

We may calculate a's power coefficient with the help of Eq. (5).

$a^0: U_0(x,t) = f(x)$

$a^1: U_1(x,t) = E^{-1}[s^\alpha E\{-H_0 + h_0\}]$

$a^2: U_2(x,t) = E^{-1}[s^\alpha E\{-H_1 + h_1\}]$

Similarly

$$a^n: U_n(x,t) = E^{-1}[s^\alpha E\{-H_{n-1} + h_{n-1}\}] \quad (6)$$

The analytical solution $U_n(x,t)$ is finally approximated by

$$U(x,t) = \lim_{n \to \infty} \sum_{r=0}^{n} U_n(x,t) \quad (7)$$

Solutions in series to the aforementioned equation converge quickly.

THE FRACTIONAL FOKKER-PLANCK EQUATION IN NONLINEAR SPACE TIME AND ITS SOLUTION VIA ETHPM

In the next part, we will show the ETHPM techniques using a variety of different specific examples. Because the accurate solution for the rare case $\alpha = 1$ is already known in advance, the beginning and boundary conditions for these cases are derived directly from this solution. This is because the precise solution for the case is already known in advance. Nevertheless, it is vital to apply this method in order to assess the accuracy of the analytical procedures and investigate how shifting the order of the space- and time-fractional derivatives influences the dynamics of the solution. MATLAB, which is a program for symbolic calculus, was utilized in order to acquire each and every one of the results.

Problem 1: Take the fractional nonlinear space-time Fokker-Planck equation as an example.

$$\frac{\partial^\alpha U}{\partial t^\alpha} = \left[\frac{\partial^\beta}{\partial x^\beta}\left(\frac{6U}{x} + 5x\right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}} U\right] U(x,t), \alpha > 0, \beta < 1 \quad (8)$$

With the starting point in mind $U(x,0) = x^3$ (9)

When we apply the procedure outlined in Section 2, we are able to obtain the coefficient of powers of α.

$a^0: U_0(x,t) = x^3$

$$a^1: U_1(x,t) = \left[-\frac{720}{\Gamma 6-\beta} x^{5-\beta} - \frac{120}{\Gamma(5-\beta)} x^{4-\beta} + \frac{720}{\Gamma(7-2\beta)} x^{6-2\beta}\right] \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$a^2: U_2(x,t) = \left[\frac{8640}{\Gamma(6-\beta)} \frac{\Gamma(8-\beta)}{\Gamma(8-2\beta)} x^{7-2\beta} + \frac{1440}{\Gamma(7-2\beta)} \frac{\Gamma(10-2\beta)}{\Gamma(10-4\beta)} x^{9-4\beta}\right] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Nonlinear Fokker-Planck equation is numerically solved as

$$u(x,t) = u_0 + u_1 + u_2 = 3 + \left[-\frac{720}{\Gamma 6-\beta} x^{5-\beta} - \frac{120}{\Gamma(5-\beta)} x^{4-\beta} + \frac{720}{\Gamma(7-2\beta)} x^{6-2\beta}\right] \frac{t^\alpha}{\Gamma(\alpha+1)} + \left[\frac{8640}{\Gamma(6-\beta)} \frac{\Gamma(8-\beta)}{\Gamma(8-2\beta)} x^{7-2\beta} + \frac{1440}{\Gamma(7-2\beta)} \frac{\Gamma(10-2\beta)}{\Gamma(10-4\beta)} x^{9-4\beta}\right] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \quad (10)$$

For $\alpha = 1, \beta = 1$ The nonlinear FPE has an exact solution as

$$U(x,t) = x^3 e^{-20t} \quad (11)$$

Problem 3.2: Take the fractional nonlinear space-time FPE as an example

$$\frac{\partial^\alpha U}{\partial t^\alpha} = \left[\frac{\partial^\beta}{\partial x^\beta}\left(\frac{3}{4}\frac{U}{x} - \frac{2x}{5}\right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}} U\right] U(x,t), \alpha > 0, \beta < 1 \quad (12)$$

With the starting point in mind $u(x,0) = x^{3/8}$ (13)

When we apply the procedure outlined in Section 2, we are able to obtain the coefficient of powers of α.

$a^0: U_0(x,t) = x^{3/8}$

$$a^1: U_1(x,t) = \left[-\frac{3}{4}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\frac{3}{4}-\beta} x^{-1/4-\beta} + \frac{2}{5}\frac{\Gamma\left(\frac{19}{8}\right)}{\Gamma\left(\frac{19}{8}-\beta\right)} x^{11/8-\beta} + \frac{\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{7}{4}-2\beta\right)} x^{\frac{3}{4}-2\beta}\right] \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$a^2: U_2(x,t) = \left[\frac{9}{8}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{3}{4}-\beta\right)} \frac{\Gamma\left(\frac{1}{8}-\beta\right)}{\Gamma\left(\frac{1}{8}-2\beta\right)} x^{-7/8-2\beta} + 2\frac{\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{7}{4}-2\beta\right)} \frac{\Gamma\left(\frac{17}{8}-2\beta\right)}{\Gamma\left(\frac{17}{8}-4\beta\right)} x^{9/8-4\beta}\right] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Nonlinear FPE is numerically solved as

$$U(x,t) = x^{3/8} + \left[-\frac{3}{4}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\frac{3}{4}-\beta} x^{-1/4-\beta} + \frac{2}{5}\frac{\Gamma\left(\frac{19}{8}\right)}{\Gamma\left(\frac{19}{8}-\beta\right)} x^{11/8-\beta} + \frac{\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{7}{4}-2\beta\right)} x^{\frac{3}{4}-2\beta}\right] \frac{t^\alpha}{\Gamma(\alpha+1)} + \left[\frac{9}{8}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{3}{4}-\beta\right)} \frac{\Gamma\left(\frac{1}{8}-\beta\right)}{\Gamma\left(\frac{1}{8}-2\beta\right)} x^{-7/8-2\beta} + 2\frac{\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{7}{4}-2\beta\right)} \frac{\Gamma\left(\frac{17}{8}-2\beta\right)}{\Gamma\left(\frac{17}{8}-4\beta\right)} x^{9/8-4\beta}\right] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \quad (14)$$

For $\alpha = 1, \beta = 1$ The nonlinear FPE has an exact solution as

$$U(x,t) = x^{3/8} e^{-\frac{2}{5}\left(\frac{3}{8}+1\right)t} = x^{3/8} e^{-11/20t} \quad (15)$$

Problem 3.3: Consider Take the fractional nonlinear space-time Fokker-Planck equation as an example

$$\frac{\partial^\alpha U}{\partial t^\alpha} = \left[\frac{\partial^\beta}{\partial x^\beta}\left(\frac{5}{2}\frac{U}{x}\right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}} U\right] U(x,t), \alpha > 0, \beta < 1 \quad (16)$$

With the starting point in mind $U(x,0) = x^{5/4}$ (17)

When we apply the procedure outlined in Section 2, we are able to obtain the coefficient of powers of α.

$a^0: U_0(x,t) = x^{5/4}$

$$a^1: U_1(x,t) = \left[-\frac{5}{2}\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{5}{2}-\beta\right)} x^{a-1-\beta} + \frac{\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{7}{2}-2\beta\right)} x^{a-2\beta}\right] \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$a^2: U_2(x,t) = \left[\left(\frac{25}{2}\right)\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{5}{2}-\beta\right)} \frac{\Gamma\left(\frac{11}{4}-\beta\right)}{\Gamma\left(\frac{11}{4}-2\beta\right)} x^{\frac{7}{4}-2\beta} + 2\frac{\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\left(\frac{7}{2}\right)-2\beta\right)} \frac{\Gamma\left(\frac{19}{4}-2\beta\right)}{\Gamma\left(\frac{19}{4}-4\beta\right)} x^{\frac{15}{4}-4\beta}\right] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Nonlinear FPE is numerically solved as

$u(x,t) =$

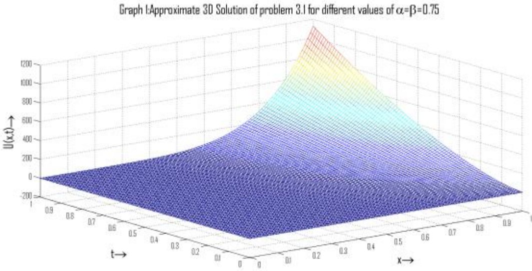
$$x^{5/4} + \left[-\frac{5}{2}\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{5}{2}-\beta\right)}x^{\alpha-1-\beta} + \frac{\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{7}{2}-2\beta\right)}x^{\alpha-2\beta}\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)} +$$

$$\left[\left(\frac{25}{2}\right)\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{5}{2}-\beta\right)}\frac{\Gamma\left(\frac{11}{4}-\beta\right)}{\Gamma\left(\frac{11}{4}-2\beta\right)}x^{\frac{7}{4}-2\beta} + 2\frac{\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\left(\frac{7}{2}\right)-2\beta\right)}\frac{\Gamma\left(\frac{19}{4}-2\beta\right)}{\Gamma\left(\frac{19}{4}-4\beta\right)}x^{\frac{15}{4}-4\beta}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

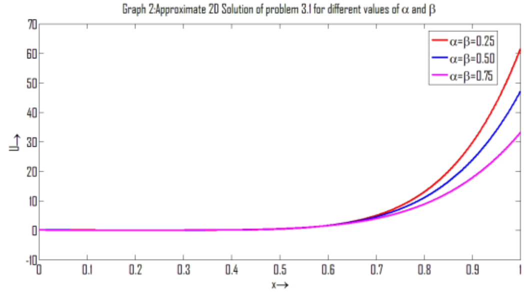
For $\alpha = 1, \beta = 1$ The nonlinear FPE has an exact solution as$u(x,t) = x^{5/4}$

NUMERICAL RESULTS AND DISCUSSION

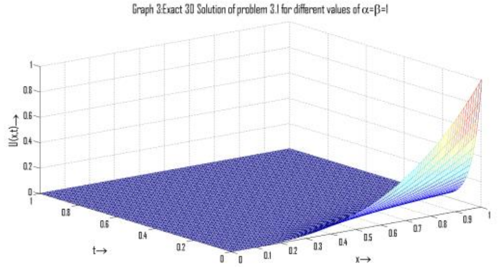
Graph 1:Approximate 3D Solution of problem 3.1 for different values of α=β=0.75



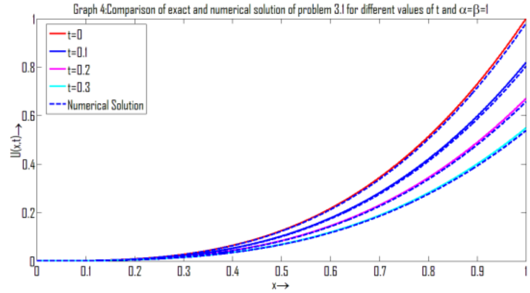
Graph 2:Approximate 2D Solution of problem 3.1 for different values of α and β



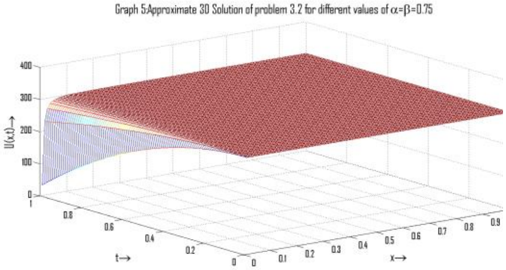
Graph 3:Exact 3D Solution of problem 3.1 for different values of α=β=1



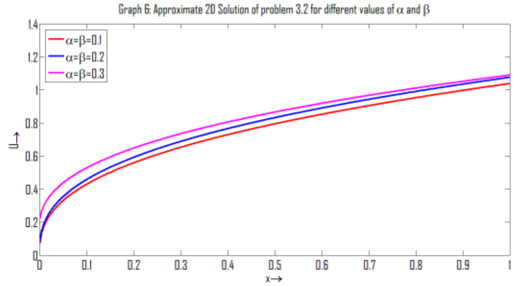
Graph 4:Comparison of exact and numerical solution of problem 3.1 for different values of t and α=β=1



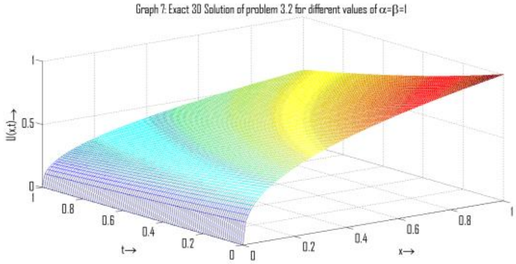
Graph 5:Approximate 3D Solution of problem 3.2 for different values of α=β=0.75



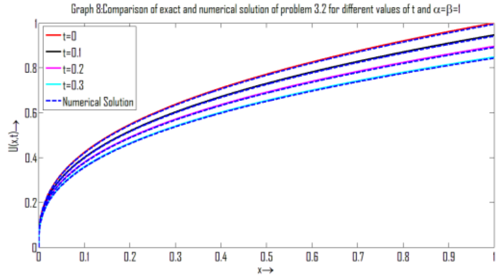
Graph 6: Approximate 2D Solution of problem 3.2 for different values of α and β

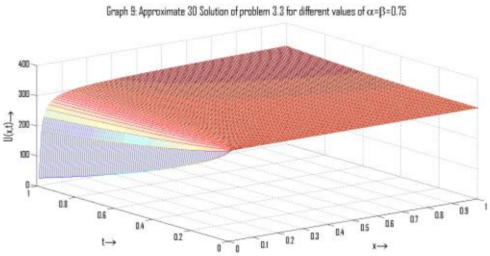


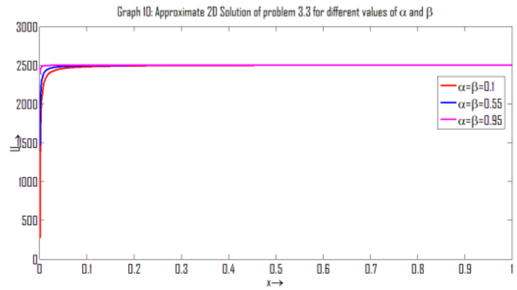
Graph 7: Exact 3D Solution of problem 3.2 for different values of α=β=1



Graph 8:Comparison of exact and numerical solution of problem 3.2 for different values of t and α=β=1







CONCLUDING REMARKS

In this work, we introduce ETHPM as a new method for addressing the Fokker-Planck problem. Numerical evidence demonstrates that the proposed approach successfully locates precise and approximate solutions for FPE. Four separate problems were used to confirm the method's effectiveness and approximation. It's also worth noting that ETHPM, in comparison to more conventional methods, can drastically cut down on computing time while still producing accurate numerical results. The proposed method can deal with non-linear situations without resorting to Adomian polynomials, which is a significant benefit over the decomposition method. Finally, this method can be applied to various FPE problems, both linear and non-linear.

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