CHAPTER: 16

REDUCED DIFFERENTIAL TRANSFORM METHOD FOR TIME FRACTIONAL NONLINEAR PDES: MATHEMATICAL FRAMEWORK AND COMPUTATIONAL ASPECTS

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ABSTRACT

Because of its computing efficiency, precision, and relative simplicity, the Reduced Differential Transform Method (RDTM) has become a powerful analytical tool for solving many types of partial differential equations (PDEs). Our focus here is on time fractional nonlinear PDEs, a large and important class of mathematical models, and how RDTM may be applied to solve them. Traditional solution methods are challenged by the complicated behaviour of these equations, which emerge in different scientific areas including physics, engineering, and biology. This study lays out a full mathematical scheme for solving time fractional nonlinear PDEs using RDTM. First, we provide an overview of RDTM and how it has been modified to deal with fractional derivatives. Afterwards, we describe in detail the process for applying RDTM to nonlinear PDEs with fractional temporal derivatives, elaborating on important computational details and algorithmic stages. We show numerical examples of the methodology in action, showing that it efficiently and accurately solves various time fractional nonlinear PDE types. In addition, we go over the computational details of using RDTM to solve time fractional nonlinear PDEs. In sum, the paper meticulously lays out the theoretical groundwork and computational details of using RDTM to solve time fractional nonlinear PDEs. The offered methodology provides a useful resource for academics and professionals in quest of effective answers to complicated mathematical models found in many scientific and engineering domains. Keywords: Mathematical Framework, Fractional Derivatives, Analytical Solution, Efficiency.

Keywords: *Reduced Differential Transform Method, Time Fractional Nonlinear PDES, Mathematical Framework, Computational Aspects*

1. INTRODUCTION

In many branches of science, including engineering, biology, physics, and finance, partial differential equations (PDEs) are crucial for the modelling of diverse physical events. A large number of real-world systems display complicated dynamics and anomalous diffusion behaviour, which has piqued the interest of many different types of partial differential equations (PDEs). One such type is the fractional temporal derivative PDE. Time fractional nonlinear partial differential equations (PDEs) are notoriously difficult to solve using standard analytical and numerical methods due to the inherent difficulties in dealing with nonlinearity and fractional order.

The Reduced Differential Transform Method (RDTM) is a relatively new yet very effective analytical approach for solving ordinary and partial differential equations. When dealing with complicated mathematical models, RDTM is a good choice because of its many benefits, including as its correctness, simplicity, and computing efficiency. To avoid iterative numerical techniques, RDTM reduces the issue dimensionality through a transformation process, which allows for the creation of analytical or semi-analytical solutions.

This work primarily addresses the usage of RDTM in this setting to solve the difficult class of time fractional nonlinear PDEs. A potential strategy for solving these equations accurately, which would allow us to grasp the underlying dynamics and phenomena better, is the combination of RDTM with fractional calculus. In order to guarantee efficient and practical implementation of RDTM, this study aims to develop a complete mathematical framework for time fractional nonlinear PDEs. It will also cover relevant computational problems.

For the purpose of solving linear and nonlinear Klein–Gordon equations, Kant and Aruna (2009) created a relatively new method of solution known as the differential transform method. This method is an exact series method of solution. Following the introduction of an operational matrix for the purpose of representing the fractional derivative in the Caputo sense, Lakestani et al. (2012) utilised this matrix in order to solve the fractional partial differential equation.

For the purpose of solving time-fractional fourth order parabolic partial differential equations, Sohail and Mohyud-Din (2012) utilised the Reduced Differential Transform Method (RDTM). The solution to FPDEs was obtained by Babolian et al. (2014) by the utilisation of a mix of the ADM and the Spectral scheme strategies.

Singh et al. (2016) have provided the analysis of coupled fractional Burger's equations by utilising a homotopy technique. This work was just published. A recently developed fractional model for tumor-immune surveillance was investigated by Baleanu et al. (2019). The model included both singular and non-singular derivative operators from the beginning.

The authors Dubey et al. (2020) expand the use of a magnificent computational methodology known as the residual power series method (RPSM) in order to obtain fractional power series solutions for non-homogeneous and homogeneous nonlinear time-fractional systems of partial differential equations. Using two advanced processes, namely the q-homotopy analysis transform method and the residual power series method, Khan et al. (2022) were able to solve the higher nonlinear issues of fractional advection-diffusion equations and systems of nonlinear fractional Burger's equations. This was accomplished by solving the equations. For the purpose of solving the time fractional partial differential equations, Jebreen and Cattani (2022) suggested a numerical technique that was characterised by its foundation in the Galerkin method. The study of time-fractional nonlinear ffth-order Korteweg–de Vries equations was provided by Prakasha et al. (2023). This was accomplished through the utilisation of an appropriate innovative technique, with the q-homotopy analysis transform method being the method in question.

In the remaining sections of this work, the following structure is used: RDTM and its expansion to handle fractional derivatives are discussed in Section 2, which offers a concise summary of the theoretical foundations of RDTM. In Section 3, we offer numerical examples to highlight the usefulness of the proposed methodology in generate correct solutions for a variety of time fractional nonlinear PDEs. These examples are included in order to demonstrate the effectiveness of the methodology. Conclusions are presented in Sections (4) and (5), which contain the results, discussion, and concluding notes. We hope that by conducting this research, we will be able to make a contribution to the development of quantitative and computational methods for solving time fractional nonlinear partial differential equations (PDEs), which will in turn make it easier to investigate and comprehend complicated dynamical systems in a variety of scientific fields.

2. BASIC IDEA OF REDUCED DIFFERENTIAL TRANSFORM METHOD

A Taylor series solution of differential equations can be obtained through the use of the reduced differential transforms technique, which is an iterative procedure. This method is easily adaptable to a wide variety of nonlinear physical situations, and it decreases the amount of computational work that needs to be done.

For the function $\theta(\lambda,\mu)$, the differential transform is defined by the following equation:

$$\phi_k(\lambda) = \frac{1}{k!}\left[\frac{\partial^k}{\partial\lambda^k}\theta(\lambda,\mu)\right]_{\mu=0} \quad (1)$$

Table 1: Reduced differential transform of some basic functionals

S.No.	Functional Form	Transformed Form
1.	$\theta(\lambda,\mu) \pm \omega(\lambda,\mu)$	$\phi_k(\lambda) \pm \psi_k(\lambda)$
2.	$d\,\theta(\lambda,\mu)$	$d\,\phi_k(\lambda)$, where d is the constant.
3.	$\theta(\lambda,\mu) = \lambda^m\mu^n$	$\phi_k(\lambda) = \lambda^m\delta(k-n)$
4.	$\lambda^m\mu^n\theta(\lambda,\mu)$	$\phi_k(\lambda) = \lambda^m\phi_{k-n}(\lambda)$
5.	$\theta(\lambda,\mu)\omega(\lambda,\mu)$	$\sum_{r=0}^{k}\phi_r(\lambda)\,\psi_{k-r}(\lambda)$
6.	$\frac{\partial^r}{\partial\lambda^r}\theta(\lambda,\mu)$	$\frac{k+r!}{k!}\phi_{k+r}(\lambda)$
7.	$\frac{\partial}{\partial\lambda}\theta(\lambda,\mu)$	$\frac{\partial}{\partial\lambda}\phi_k(\lambda)$
8.	$\frac{\partial^2}{\partial\lambda^2}\theta(\lambda,\mu)$	$\frac{\partial^2}{\partial\lambda^2}\phi_k(\lambda)$

3. NUMERICAL ILLUSTRATIONS

For the purpose of demonstrating how the Reduced Differential Transform Method (RDTM) can be utilized to solve a time fractional nonlinear partial differential equation (PDE), let us have a look at a specific case.

Illustration 3.1: $\frac{\partial^\epsilon\theta}{\partial\mu^\epsilon} = \frac{\partial^2\theta}{\partial\lambda^2} + \theta^2; 1 \le \epsilon \le 2, \mu > 0$ (2)

with initial condition $\theta(\lambda,0) = 1 + sin\lambda$ (3)

$$\frac{\Gamma(k\epsilon+\epsilon+1)}{\Gamma(k\epsilon+1)}\phi_{k+1}(\lambda) = \frac{\partial^2}{\partial\lambda^2}\phi_k(\lambda) + \sum_{r=0}^{k}\phi_r(\lambda)\phi_{k-r}(\lambda)$$

$\phi_0(\lambda) = 1 + sin\lambda$ (4)

$k = 0$

$\frac{\Gamma(\epsilon+1)}{\Gamma 1}\phi_1(\lambda) = \frac{\partial^2}{\partial\lambda^2}\phi_0(\lambda) + \sum_{r=0}^{0} \quad \phi_r(\lambda)\phi_{0-r}(x)$

$\Gamma(\epsilon+1)\phi_1(\lambda) = \frac{\partial^2}{\partial\lambda^2}(1 + sin\lambda) + \phi_0(\lambda)\phi_0(\lambda)$

$\Gamma(\epsilon+1)\phi_1(\lambda) = -sin\lambda + (1 + sin\lambda)^2 = -sin\lambda + 1 + sin^2\lambda + 2sin\lambda = 1 + sin^2\lambda + sin\lambda$

$\phi_1(\lambda) = \frac{1}{\Gamma(\epsilon+1)}(1 + sin^2\lambda + sin\lambda\ )$ (5)

$k = 1$

$\frac{\Gamma(2\epsilon+1)}{\Gamma(\epsilon+1)}\phi_2(\lambda) = \frac{\partial^2}{\partial\lambda^2}\phi_1(\lambda) + \sum_{r=0}^{1} \quad \phi_r(\lambda)\phi_{1-r}(\lambda)$

$\frac{\Gamma(2\epsilon+1)}{\Gamma(\epsilon+1)}\phi_2(x) = \frac{\partial^2}{\partial\lambda^2}\phi_1(x) + [\phi_0(x)\phi_1(x) + \phi_1(x)\phi_0(x)]$

$\frac{\Gamma(2\epsilon+1)}{\Gamma(\gamma+1)}\phi_2(\lambda) = \frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{\Gamma(\gamma+1)}(1 + sin^2x + sinx\ )\right] + \frac{1}{\Gamma(\gamma+1)}(1 + sinx\ )(1 + sin^2x + sinx\ )$

$\Gamma(2\epsilon+1)\phi_2(\lambda) = 2cos^2\lambda - 2sin^2\lambda - sin\lambda + 1 + sin^2\lambda + sin\lambda + sin\lambda + sin^3\lambda + sin^2\lambda$

$\Gamma(2\epsilon+1)\phi_2(\lambda) = 1 + sin\lambda + sin^3\lambda + 2cos^2\lambda$

$\phi_2(\lambda) = \frac{1}{\Gamma(2\epsilon+1)}(1 + sin\lambda + sin^3\lambda + 2cos^2\lambda)$ (6)

$\theta(\lambda,\mu) = \sum_{k=0}^{\infty} \quad \phi_k(\lambda)\mu^{\epsilon k} = \phi_0(\lambda) + \phi_1(\lambda)\mu^{\epsilon} + \phi_2(\lambda)\mu^{2\epsilon} + \cdots$

$\theta(\lambda,\mu) = 1 + sin\lambda + \frac{1}{\Gamma(\epsilon+1)}(1 + sin^2\lambda + sin\lambda\ )\mu^{\epsilon} + \frac{1}{\Gamma(2\epsilon+1)}(1 + sin\lambda + sin^3\lambda + 2cos^2\lambda)\mu^{2\epsilon} + \cdots$ (7)

Illustration 3.2: Taking into consideration the nonlinear time-fractional equation that follows

$\frac{\partial^{\epsilon}\theta}{\partial\mu^{\epsilon}} + \theta\frac{\partial\theta}{\partial\lambda} = \lambda + \lambda\mu^2; 0 < \epsilon \leq 1, t > 0$ (8)

with initial condition $u(\lambda, 0) = \lambda^2$ (9)

$\frac{\Gamma(k\epsilon+\gamma+1)}{\Gamma(k\epsilon+1)}\phi_{k+1}(x) + \sum_{r=0}^{k} \quad \phi_r(\lambda)\frac{\partial}{\partial x}\phi_{k-r}(\lambda) = \lambda\delta(k) + \lambda\delta(k-2)$

$\phi_0(\lambda) = \lambda^2$ (10)

$k = 0$

$\frac{\Gamma(\epsilon+1)}{\Gamma 1}\phi_1(\lambda) + \sum_{r=0}^{0} \quad \phi_r(\lambda)\frac{\partial}{\partial\lambda}\phi_{0-r}(\lambda) = \lambda\delta(0) + \lambda\delta(0-2)$

$\frac{\Gamma(\epsilon+1)}{\Gamma 1}\phi_1(\lambda) + \phi_0(\lambda)\frac{\partial}{\partial\lambda}\phi_0(\lambda) = \lambda\delta(0) + \lambda\delta(-2)$

$\Gamma(\epsilon+1)\phi_1(\lambda) + 2\lambda^3 = \lambda$

$\phi_1(\lambda) = \frac{(\lambda-2\lambda^3)}{\Gamma(\epsilon+1)}$ (11)

$k = 1$

$\frac{\Gamma(2\epsilon+1)}{\Gamma(\epsilon+1)}\phi_2(\lambda) + \sum_{r=0}^{1} \quad \phi_r(\lambda)\frac{\partial}{\partial x}\phi_{1-r}(\lambda) = \lambda\delta(1) + \lambda\delta(1-2)$

$$\frac{\Gamma(2\epsilon+1)}{\Gamma(\epsilon+1)}\phi_2(x)+\phi_0(\lambda)\frac{\partial}{\partial x}\phi_1(\lambda)+\phi_1(x)\frac{\partial}{\partial x}\phi_0(\lambda)=\lambda\delta(1)+\lambda\delta(-1)$$

$$\frac{\Gamma(2\epsilon+1)}{\Gamma(\epsilon+1)}\phi_2(\lambda)+\lambda^2\left[\frac{1-6\lambda^2}{\Gamma(\epsilon+1)}\right]+\frac{2\lambda^3-4\lambda^4}{\Gamma(\epsilon+1)}=0$$

$$\phi_2(\lambda)=\frac{(10\lambda^4-2\lambda^3-\lambda^2)}{\Gamma(2\epsilon+1)} \tag{12}$$

$$\theta(\lambda,\mu)=\sum_{k=0}^{\infty}\phi_k(\lambda)\mu^{\epsilon k}=\lambda^2+\frac{(\lambda-2\lambda^3)}{\Gamma(\epsilon+1)}\mu^{\epsilon}+\frac{(10\lambda^4-2\lambda^3-\lambda^2)}{\Gamma(2\epsilon+1)}\mu^{2\epsilon}+\cdots \tag{13}$$

Illustration 3.3: $\frac{\partial^{\epsilon}\theta}{\partial\mu^{\epsilon}}+u\frac{\partial\theta}{\partial\lambda}=2\mu+\lambda+\mu^3+\lambda\mu^2;0<\epsilon\le 1,\mu>0$ (14)

with initial condition $u(\lambda,0)=e^{\lambda}$ (15)

$$\frac{\Gamma(k\epsilon+\epsilon+1)}{\Gamma(k\epsilon+1)}\phi_{k+1}(\lambda)+\sum_{r=0}^{k}\phi_r(\lambda)\frac{\partial}{\partial\lambda}\phi_{k-r}(\lambda)=2\delta(k-1)+\lambda\delta(k)+\delta(k-3)+\lambda\delta(k-2)$$

$$\phi_0(\lambda)=e^{\lambda} \tag{16}$$

$k=0$

$$\frac{\Gamma(\epsilon+1)}{\Gamma 1}\phi_1(\lambda)+\sum_{r=0}^{0}\phi_r(\lambda)\frac{\partial}{\partial\lambda}\phi_{0-r}(\lambda)=2\delta(-1)+\lambda\delta(0)+\delta(-3)+\lambda\delta(-2)$$

$$\Gamma(\epsilon+1)\phi_1(\lambda)+\phi_0(\lambda)\frac{\partial}{\partial\lambda}\phi_0(\lambda)=\lambda$$

$$\Gamma(\epsilon+1)\phi_1(\lambda)+e^{2\lambda}=\lambda$$

$$\phi_1(\lambda)=\frac{\lambda-e^{2\lambda}}{\Gamma(\epsilon+1)} \tag{17}$$

$k=1$

$$\frac{\Gamma(2\epsilon+1)}{\Gamma(\epsilon+1)}\phi_2(\lambda)+\sum_{r=0}^{1}\phi_r(\lambda)\frac{\partial}{\partial\lambda}\phi_{1-r}(\lambda)=2\delta(0)+\lambda\delta(1)+\delta(-2)+\lambda\delta(-1)$$

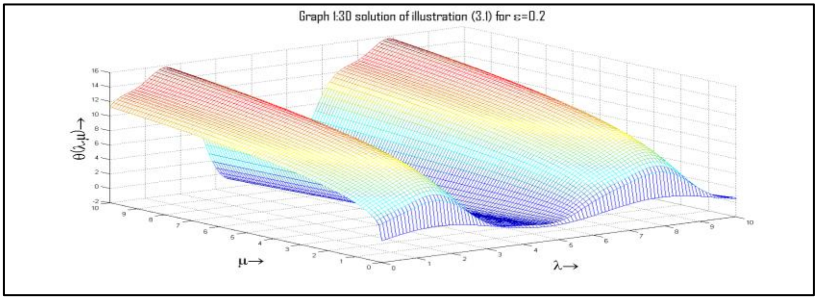
$$\frac{\Gamma(2\epsilon+1)}{\Gamma(\epsilon+1)}\phi_2(\lambda)+\phi_0(\lambda)\frac{\partial}{\partial\lambda}\phi_1(x)+\phi_1(\lambda)\frac{\partial}{\partial x}U_0(\lambda)=2$$

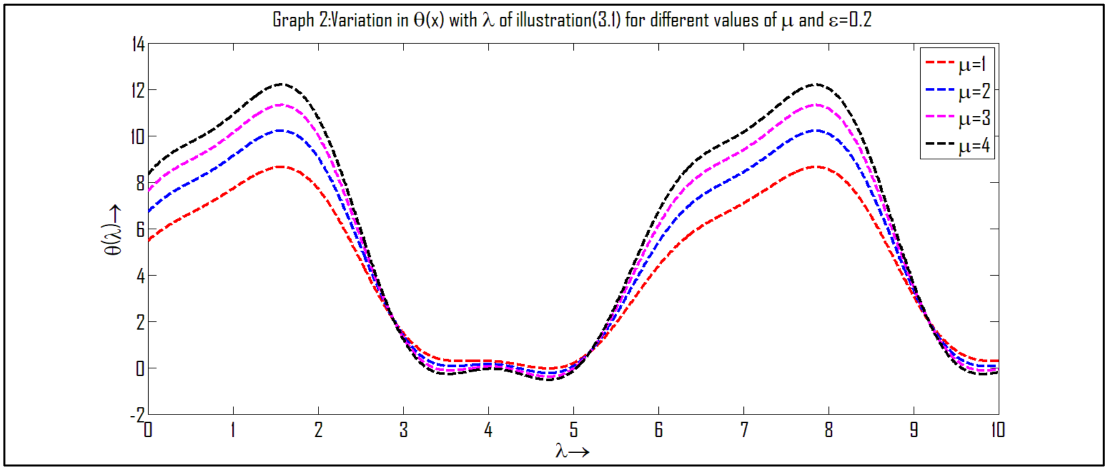
$$\frac{\Gamma(2\epsilon+1)}{\Gamma(\epsilon+1)}\phi_2(\lambda)+e^{\lambda}\left[\frac{1-2e^{2\lambda}}{\Gamma(\gamma+1)}\right]+\left[\frac{\lambda-e^{2\lambda}}{\Gamma(\gamma+1)}\right]e^{\lambda}=2$$

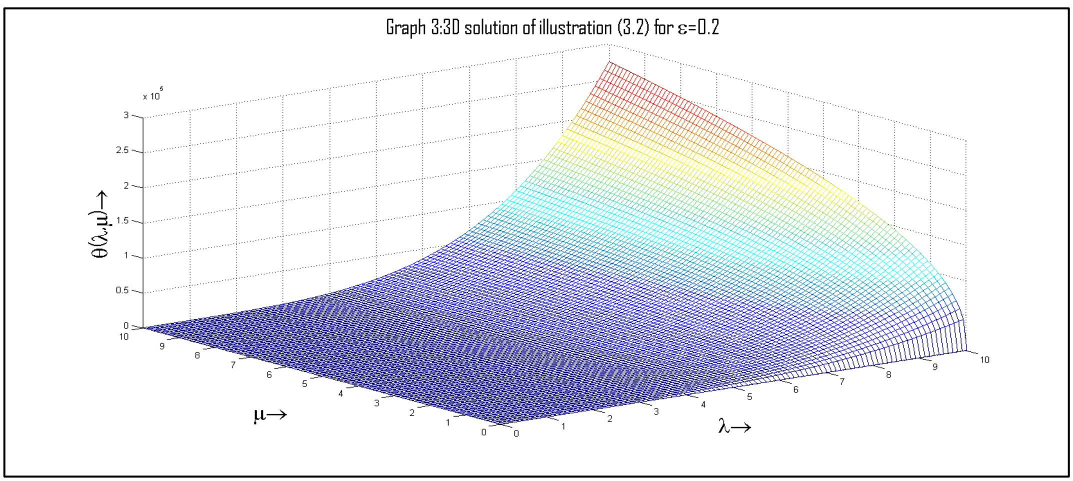
$$\phi_2(\lambda)=\frac{1}{\Gamma(2\epsilon+1)}\left[2\Gamma(\epsilon+1)+2e^{3\lambda}+e^{2\lambda}-e^{\lambda}-\lambda\right] \tag{18}$$

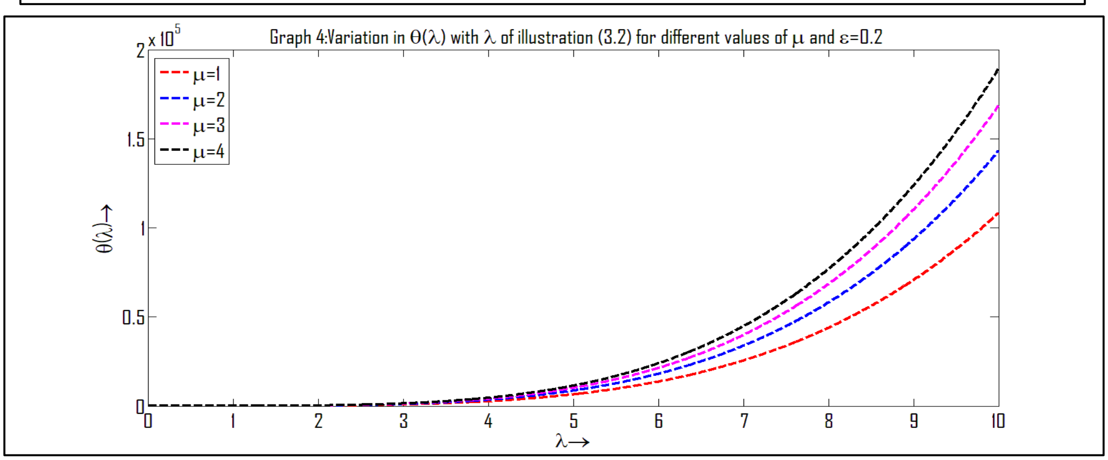
$$\theta(\lambda,\mu)=e^{\epsilon}+\frac{\lambda-e^{2\lambda}}{\Gamma(\epsilon+1)}\mu^{\epsilon}+\frac{1}{\Gamma(2\epsilon+1)}\left[2\Gamma(\epsilon+1)+2e^{3\lambda}+e^{2\lambda}-e^{\lambda}-\lambda\right]\mu^{2\epsilon}+\cdots \tag{19}$$

4. RESULTS AND DISCUSSION

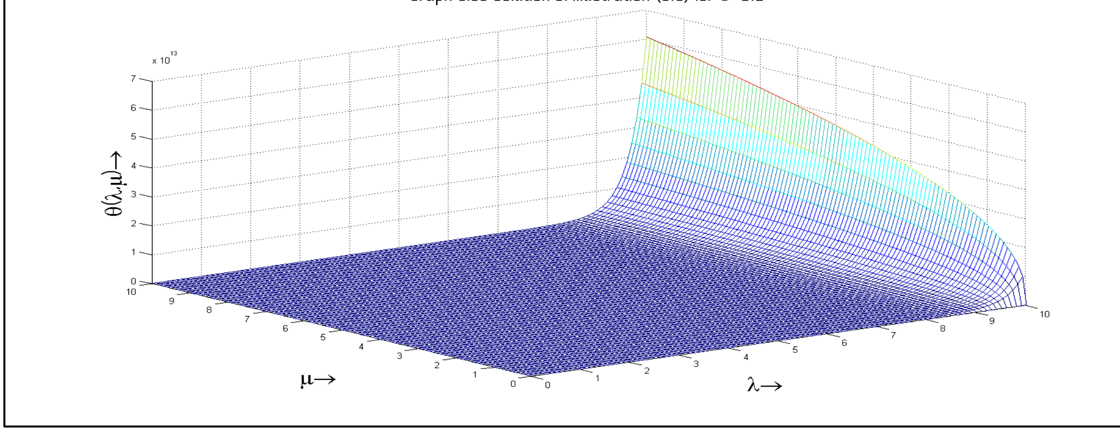



Graph 2:Variation in Θ(x) with λ of illustration(3.1) for different values of μ and ε=0.2
θ(λ)→
λ→
μ=1
μ=2
μ=3
μ=4


Graph 3:3D solution of illustration (3.2) for ε=0.2
x 10^5
θ(λ,μ)→
μ→
λ→


Graph 4:Variation in Θ(λ) with λ of illustration (3.2) for different values of μ and ε=0.2
x 10^5
θ(λ)→
λ→
μ=1
μ=2
μ=3
μ=4

Graph 5:3D solution of illustration (3.3) for ε=0.2



Graph 6:Variation in θ(λ) with λ of illustration(3.3) for different values of μ and ε=0.2

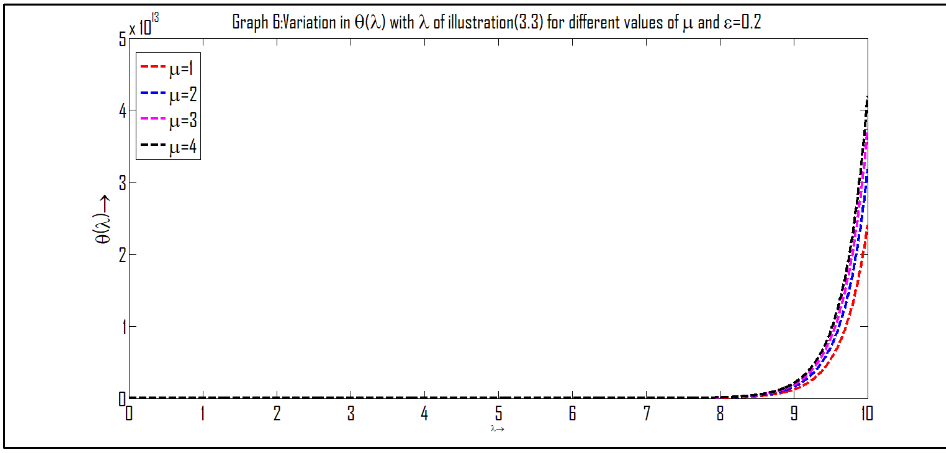


Table 2: Numerical values of the solution of illustration (3.1)				
	$\mu = 1$	$\mu = 2$	$\mu = 3$	$\mu = 4$
λ	$\theta(\lambda)$	$\theta(\lambda)$	$\theta(\lambda)$	$\theta(\lambda)$
0	5.4703	6.7126	7.6038	8.3241
1	7.7208	9.1249	10.1152	10.9072
2	7.7261	9.0756	10.0238	10.7806
3	1.5078	1.409	1.323	1.2464
4	0.2879	0.1504	0.0406	-0.0536
5	0.1912	0.0605	-0.0462	-0.1385
6	4.3733	5.3918	6.1228	6.7137
7	7.0848	8.4304	9.3837	10.1483
8	8.5487	10.0923	11.1789	12.047
9	3.1005	3.3457	3.5048	3.6254
10	0.2758	0.0783	-0.0744	-0.203

Table 3: Numerical values of the solution of illustration (3.2)				
	$\mu = 1$	$\mu = 2$	$\mu = 3$	$\mu = 4$
λ	$\theta(\lambda)$	$\theta(\lambda)$	$\theta(\lambda)$	$\theta(\lambda)$
0	0	0	0	0
1	0.0001	0.0001	0.0001	0.0001
2	0.0015	0.0019	0.0023	0.0026
3	0.008	0.0106	0.0125	0.014
4	0.026	0.0345	0.0407	0.0458
5	0.0649	0.086	0.1014	0.114
6	0.1365	0.1808	0.2131	0.2394
7	0.2554	0.3381	0.3984	0.4475
8	0.439	0.5809	0.6842	0.7685
9	0.7071	0.9355	1.1018	1.2373
10	1.0827	1.432	1.6863	1.8935

Table 4: Numerical values of the solution of illustration (3.3)				
	$\mu = 1$	$\mu = 2$	$\mu = 3$	$\mu = 4$
λ	$\theta(\lambda)$	$\theta(\lambda)$	$\theta(\lambda)$	$\theta(\lambda)$
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
5	0	0	0	0
6	0	0	0	0
7	0.0003	0.0004	0.0005	0.0005
8	0.006	0.0079	0.0093	0.0104
9	0.1199	0.1583	0.1861	0.2088
10	2.4089	3.1785	3.7382	4.1941

The 3D plots depicting the solutions of examples (3.1), (3.2), and (3.3) are presented in graphs (1), (3), and (5) correspondingly. For numerical illustration (3.1), The variation in $\theta(\lambda)$ with spatial coordinate λ for the temporal coordinates $\mu = 1,2,3$ and 4 and fractional order $\epsilon = 0.2$ has been depicted in graph (2). The variation in $\theta(\lambda)$ with spatial coordinate Λ for the temporal coordinates $\mu = 1,2,3$ and 4 and fractional order $\epsilon = 0.2$ has been shown in graph (4) for numerical demonstration (3.2). It can be observed that when λ and μ grow, $\theta(\lambda)$ also grows. Graph (6) illustrates the variation in $\theta(\lambda)$ with respect to spatial coordinate λ and fractional order $\epsilon = 0.2$ for temporal coordinates $\mu = 1,2,3$ and 4. This information is provided for numerical illustration (3.3). It is observed that as λ and μ increase, so does $\theta(\lambda)$.

5. CONCLUDING REMARKS

Finally, a strong theoretical basis and certain useful computational features are provided by the Reduced Differential Transform Method (RDTM), which is an attractive strategy for solving time fractional nonlinear PDEs. Reducing the dimensionality of the problem and providing analytical or semi-analytical solutions, RDTM first transforms the original PDE into a collection of algebraic equations, solves them, and then inversely transforms the result back to the original domain. Time fractional nonlinear PDEs efficiently handle nonlinearity and fractional derivatives thanks to the mathematical framework of RDTM. In order to understand how complicated dynamical systems behave, RDTM uses systematic transformation and solution approaches to make it easier to derive solutions in terms of auxiliary functions or series expansions. Together, RDTM's computational features and mathematical foundation make it an effective tool for solving time fractional nonlinear PDEs. Researchers in many fields of science and engineering benefit from RDTM's accurate and efficient solutions as they investigate and understand complicated dynamical processes. Further improvements in analytical and computational methods for solving difficult PDEs may be possible with ongoing research and development in this field.

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