CHAPTER: 30

A DOUBLE INTERPOLATION APPROACH TO APPROXIMATE HEAT CONDUCTION EQUATION WITH SOURCE/SINK TERM

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ABSTRACT

This paper proposes a novel double interpolation approach to approximate the heat conduction equation with a source/sink term. The heat conduction equation is a fundamental partial differential equation governing the transfer of thermal energy in various physical systems. In many practical scenarios, source or sink terms are present, representing external heat generation or absorption. Traditional numerical methods may struggle to accurately capture the behavior of such systems. This approach enhances the accuracy of the approximation by effectively capturing the spatial and temporal variations of the temperature distribution. We demonstrate the effectiveness of our approach through numerical experiments and comparisons with exact solution. The results show that the double interpolation method offers significant improvements in accuracy and computational efficiency particularly when dealing with complex heat conduction problems involving source or sink terms. Overall, this study contributes to advancing the numerical approximation techniques for heat conduction equations with practical implications in various fields, including thermal engineering, materials science, and environmental modeling.

Keywords: *Heat conduction equation, Source/sink term, Double interpolation, Approximation, Accuracy.*

1. INTRODUCTION

The heat conduction equation is a fundamental partial differential equation governing the transfer of thermal energy in various physical systems. It plays a crucial role in numerous fields such as engineering, physics, materials science, and environmental studies. Often, real-world scenarios involve additional complexities, such as the presence of source or sink terms representing external heat generation or absorption. Traditional numerical methods for solving the heat conduction equation may encounter challenges in accurately capturing the behavior of systems with source/sink terms.

In recent years, various numerical techniques have been developed to address these challenges. One promising approach is the use of interpolation methods, which offer a flexible and efficient way to approximate the spatial and temporal variations of the temperature field. In this paper, we propose a novel double interpolation approach to approximate the heat conduction equation with source/sink terms.

The issue of finding a parameter in an ill-posed heat equation was studied by Wang and Zheng (1999). By using Tikhonov regularization on over-specified data, they were able to derive a stable approximation to the unknown parameter, and they also provided numerical calculations to back up our approximation.

Concerning hyperbolic heat conduction, Ciegis (2009) found a solution. Their work included the development and investigation of both explicit and implicit Euler systems. Parabolic and hyperbolic heat conduction problems can be efficiently solved using the implicit Euler approach, as they also demonstrated.

A new numerical method for solving the linear transient heat conduction equation, known as the "Explicit Green's Approach" (ExGA), was introduced by Mansur et al. (2009). As a representation of the issue domain in terms of physical qualities and geometrical characteristics, the approach makes use of Green's matrix.

A numerical solution to the averaged thermal energy equation, which is based on Fourier conduction, was recently published in the literature by White (2016). The solution is suitable for three-dimensional time-dependent and steady-state analysis since it is based on non-iterative finite difference techniques that are second-order time-accurate.

For the purpose of deriving analytical and numerical solutions of heat diffusion in one-dimensional thin rods, Makhtoumi (2017) examined adaptive approaches. They demonstrated a thorough comparison study using the finite difference approach and the homotopy perturbation method.

The analytical and numerical solutions to the heat transport problem in a finite-length rod were reported by Skrzypczak and Skrzypczak (2017). The analytical solution was derived by applying the Fourier series.

The one-dimensional steady-state heat conduction issues were addressed by Reddy K T (2019) using the finite difference approach. The findings were compared with the exact solutions obtained by utilising the Resistance formula.

Lekomtsev (2020) examined a set of quasilinear parabolic equations in one dimension that include delay effects. They developed a strategy for solving these types of problems numerically.

Using the functional variation approach, Liu et al. (2022) proved that the best performance may be ensured by demonstrating the sufficient and necessary conditions for the global optimum in the heat conduction problem.

An analytical method for obtaining the temperature field in heat sinks was suggested by Correa et al. (2023). In this method, the interface contact heat fluxes on each fin are calculated by solving a 2D model for the base and a 3D model for the fins using the Classical Integral Transform Technique (CITT). The two models are then connected using Eigen function expansions.

2. HEAT CONDUCTION EQUATION WITH HEAT SOURCE/SINK TERM:

Let's take the following heat conduction equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + R(x,t) \quad (1)$$

boundary conditions: $u(0,t) = u(1,t) = 0$ (2)

with the initial condition: $u(x,0) = 2\sin \pi x + 3sin2\pi x$ (3)

and with a source/sink function $R(x,t) = 2u\sin\pi t$ (4)

A source term in a partial differential equation represents the contribution of external factors or phenomena that affect the evolution of the dependent variable u. In this case, the term $2u\sin\pi t$ represents a spatially varying external influence on the system described by u. It contributes to the rate of change of u with respect to time $\left(\frac{\partial u}{\partial t}\right)$ and can affect the distribution of u over space and time. In the context of heat conduction, it represents a spatially varying heat source or sink within the medium.

3. EXACT SOLUTION OF PROPOSED HEAT CONDUCTION EQUATION:

Let $u = XT$ where $X \equiv X(x), T \equiv T(t)$ (5)

$XT' = \alpha X''T + 2XT\sin(\pi t)$ (6)

Dividing equation (5) by XT on both sides, we get

$$\frac{XT'}{XT} = \alpha\frac{X''T}{XT} + \frac{2XT\sin\pi t}{XT}$$

$$\frac{T'}{T} = \alpha\frac{X''}{X} + 2\sin\pi t$$

$$\alpha\frac{X''}{X} = \frac{T'}{T} - 2\sin\pi t = -k^2$$

Case-I: $\alpha\frac{X''}{X} = -k^2 \Rightarrow X'' + \frac{k^2}{\alpha}X = 0 \Rightarrow X = c_1 cos\left(\frac{k}{\sqrt{\alpha}}\right)x + c_2 sin\left(\frac{k}{\sqrt{\alpha}}\right)x$ (7)

Case-II: $\frac{T'}{T} - 2\sin\pi t = -k^2\, \frac{dT}{dt} = (2\sin\pi t - k^2)T \Rightarrow \frac{dT}{T} = (2\sin\pi t - k^2)dt$

$\Rightarrow logT = 2\pi\cos\pi t - k^2 t + logc_3 \Rightarrow T = c_3\exp(2\pi\cos\pi t - k^2 t)$ (8)

$u(x,t) = XT = \left[c_1 cos\left(\frac{k}{\sqrt{\alpha}}\right)x + c_2 sin\left(\frac{k}{\sqrt{\alpha}}\right)x\right]c_3\exp(2\pi\cos\pi t - k^2 t)$ (9)

Applying first condition of equation (2) in equation (9), we get

$$u(0,t) = c_1 c_3\exp(2\pi\cos\pi t - k^2 t) = 0 \Rightarrow c_1 = 0$$

Putting the value of c_1 in equation (9), we get

$u(x,t) = c_2 c_3\exp(2\pi\cos\pi t - k^2 t) sin\left(\frac{k}{\sqrt{\alpha}}\right)x$ (10)

Applying second condition of equation (2) in equation (10), we get

$$u(x,t) = c_2 c_3\exp(2\pi\cos\pi t - k^2 t)\, sin\left(\frac{k}{\sqrt{\alpha}}\right) = 0 \Rightarrow sin\left(\frac{k}{\sqrt{\alpha}}\right) = 0 \Rightarrow \frac{k}{\sqrt{\alpha}} = n\pi$$

Putting the value of $\frac{k}{\sqrt{\alpha}}$ in equation (10), we get

$u(x,t) = c_2 c_3\exp(2\pi\cos\pi t - k^2 t) sinn\pi x$ (11)

The most general solution is

$u(x,t) = \sum_{n=1}^{\infty} b_n\exp(2\pi\cos\pi t - k^2 t) sinn\pi x$ (12)

Applying condition of equation (3) in equation (12), we get

$u(x,0) = \sum_{n=0}^{\infty} b_n\, sinn\pi x = 2e^{-2\pi} \sin \pi x + 3e^{-2\pi} sin2\pi x$

$b_1 sin\pi x + b_2 sin2\pi x + \cdots = 2e^{-2\pi} \sin \pi x + 3e^{-2\pi} sin2\pi x$

$b_1 = 2e^{-2\pi}, b_2 = 3e^{-2\pi}, b_3 = 0, \ldots$

Putting the values of $b_1, b_2, b_3 \ldots$ in equation (12), we get

$$u(x,t) = 2e^{-2\pi} \exp(2\pi \cos \pi t - \pi^2 \alpha t)\, sin\pi x + 3e^{-2\pi} \exp(2\pi \cos \pi t - 4\pi^2 \alpha t)\, sin2\pi x \tag{13}$$

Approximate solution of proposed heat conduction by double interpolation method:

Using the double interpolation method, we may now solve equation (1) in addition to conditions (2) and (3), and we obtain

The difference interval of x as 0.2, denoted as $h = 0.2$

Calculating the duration of t as a function of Bender-Schmidt

$$k = \frac{h^2}{2c^2} = \frac{(0.2)^2}{2} = 0.02 \tag{14}$$

Thus $x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8, x_5 = 1$

$t_0 = 0, t_1 = 0.02, t_2 = 0.04, t_3 = 0.06, t_4 = 0.08, t_5 = 0.1$

Once we've drawn straight lines parallel to the coordinate axis (t,x), we'll have a total of 25 mesh points.

$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + 2u\sin\pi t$

$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k}$

$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$

$\frac{u_{i,j+1} - u_{i,j}}{k} = \alpha \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + 2u_{i,j} sin\pi t_j$

$u_{i,j+1} - u_{i,j} = \frac{k\alpha}{h^2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) + 2ku_{i,j} sin\pi t_j$

$u_{i,j+1} = \left(1 - \frac{2k\alpha}{h^2} + 2ksin\pi t_j\right) u_{i,j} + \frac{k\alpha}{h^2}\left(u_{i-1,j} + u_{i+1,j}\right)$

$$u_{i,j+1} = \left(1 - 2\lambda\alpha + 2ksin\pi t_j\right) u_{i,j} + \lambda\alpha\left(u_{i-1,j} + u_{i+1,j}\right) \tag{15}$$

where

$\lambda = \frac{k}{h^2} = \frac{0.02}{(0.2)^2} = 0.5$

$\alpha = 1$

$$u_{i,j+1} = \left[1 - 2\lambda + 2ksin\pi t_j\right] u_{i,j} + \lambda\left(u_{i-1,j} + u_{i+1,j}\right) \tag{16}$$

$i = 1, j = 0$

$u_{11} = [1 - 2(0.5) + 2(0.02)sin\pi t_0]u_{10} + 0.5(u_{00} + u_{20})$

$u_{11} = 0.5(0 + 3.6655) = 1.8328$

$i = 2, j = 0$

$u_{21} = [1 - 2(0.5) + 2(0.02)sin\pi t_0]u_{20} + (0.5)(u_{10} + u_{30})$

$u_{21} = (0.5)(4.0287 + 0.1388) = \ 2.0837$

$i = 3, j = 0$

$u_{31} = [1 - 2(0.5) + 2(0.02)sin\pi t_0]u_{30} + (0.5)(u_{20} + u_{40})$

$u_{31} = (0.5)(3.6655 - 1.6776) = \ 0.9940$

$i = 4, j = 0$

$u_{41} = [1 - 2(0.5) + 2(0.02)sin\pi t_0]u_{40} + (0.5)(u_{30} + u_{50})$

$u_{41} = (0.5)(0.1388 + 0) = 0.0694$

$u_{i,j+1} = \left[1 - 2\lambda + 2ksin\pi t_j\right]u_{i,j} + \lambda\left(u_{i-1,j} + u_{i+1,j}\right)$

$i = 1, j = 1$

$u_{12} = [1 - 2(0.5) + 2ksin\pi t_1]u_{11} + 0.5(u_{01} + u_{21})$

$u_{12} = [2(0.02)\sin(0.02\pi)]u_{11} + 0.5(u_{01} + u_{21})$

$u_{12} = [0.04\sin(0.02\pi)]1.8328 + 0.5(0 + 2.0837) = 1.0465$

$i = 2, j = 1$

$u_{22} = [1 - 2(0.5) + 2(0.02)\sin(0.02\pi)]u_{21} + 0.5(u_{11} + u_{31})$

$u_{22} = [2(0.02)\sin(0.02\pi)]2.0837 + 0.5(1.8328 + 0.994) = 1.4186$

$i = 3, j = 1$

$u_{32} = [1 - 2(0.5) + 2(0.02)\sin(0.02\pi)]u_{31} + 0.5(u_{21} + u_{41})$

$u_{32} = [0.04\sin(0.02\pi)]0.994 + 0.5(2.0837 + 0.0694) = \ 1.0790$

$i = 4, j = 1$

$u_{42} = [1 - 2(0.5) + 2(0.02)\sin(0.02\pi)]u_{41} + 0.5(u_{31} + u_{51})$

$u_{42} = [0.04\sin(0.02\pi)]0.0694 + 0.5(0.994 + 0)$

$i = 1, j = 2$

$u_{13} = [1 - 2(0.5) + 2ksin\pi t_2]u_{12} + \lambda(u_{02} + u_{22})$

$u_{13} = [2(0.02)\sin(0.04\pi)]1.0465 + 0.5(0 + 1.4186) = \quad 0.7145$

$i = 2, j = 2$

$u_{23} = [1 - 2(0.5) + 2ksin\pi t_2]u_{22} + \lambda(u_{12} + u_{32})$

$u_{23} = [2(0.02)sin(0.04\pi)]1.4186 + 0.5(1.0465 + 1.079)$

$i = 3, j = 2$

$u_{33} = [1 - 2(0.5) + 2ksin\pi t_2]u_{32} + 0.5(u_{22} + u_{42})$

$u_{33} = [2(0.02)\sin(0.04\pi)]1.079 + (0.5)(1.4186 + 0.4972)$

$i = 4, j = 2$

$u_{43} = [1 - 2(0.5) + 2ksin\pi t_2]u_{42} + 0.5(u_{32} + u_{52})$

$u_{43} = [2(0.04)sin(0.04\pi)]0.4972 + 0.5(1.079 + 0) = 0.5445$

$i = 1, j = 3$

$u_{14} = [1 - 2(0.5) + 2(0.02)\sin(0.06\pi)]u_{13} + \lambda(u_{03} + u_{23})$

$u_{14} = [1 - 2(0.5) + 2(0.02)\sin(0.06\pi)]0.7145 + 0.5(0 + 1.0699) = 0.5403$

$i = 2, j = 3$

$u_{24} = [1 - 2(0.5) + 2(0.02)\sin(0.06\pi)]u_{23} + \lambda(u_{13} + u_{33})$

$u_{24} = [1 - 2(0.5) + 2(0.02)\sin(0.06\pi)]1.0699 + 0.5(0.7145 + 0.9633)$

$i = 3, j = 3$

$u_{34} = [1 - 2(0.5) + 2(0.02)\sin(0.06\pi)]u_{33} + \lambda(u_{23} + u_{43})$

$u_{34} = [1 - 2(0.5) + 2(0.02)\sin(0.06\pi)]0.9633 + 0.5(1.0699 + 0.5445) = \quad 0.8144$

$i = 4, j = 3$

$u_{44} = [1 - 2(0.5) + 2(0.02)\sin(0.06\pi)]u_{43} + \lambda(u_{33} + u_{53})$

$u_{44} = [1 - 2(0.5) + 2(0.02)\sin(0.06\pi)]0.5445 + 0.5(0.9633 + 0) = 0.4857$

$u_{i,j+1} = \left[1 - 2\lambda + 2ksin\pi t_j\right]u_{i,j} + \lambda\left(u_{i-1,j} + u_{i+1,j}\right)$

$i = 1, j = 4$

$u_{15} = [1 - 2(0.5) + 2(0.02)\sin(0.08\pi)]u_{14} + \lambda(u_{04} + u_{24})$

$u_{15} = [1 - 2(0.5) + 2(0.02)\sin(0.08\pi)]0.5403 + 0.5(0 + 0.8469) = \ 0.4288$

$i = 2, j = 4$

$u_{25} = [1 - 2(0.5) + 2(0.02)\sin(0.08\pi)]u_{24} + \lambda(u_{14} + u_{34})$

$u_{25} = [1 - 2(0.5) + 2(0.02)\sin(0.08\pi)]0.8469 + 0.5(0.5403 + 0.8144) = 0.6858$

$i = 3, j = 4$

$u_{35} = [1 - 2(0.5) + 2(0.02)\sin(0.08\pi)]u_{34} + \lambda(u_{24} + u_{44})$

$u_{35} = [1 - 2(0.5) + 2(0.02)\sin(0.08\pi)]0.8144 + 0.5(0.8469 + 0.4857) = 0.6744$

$i = 4, j = 4$

$u_{45} = [1 - 2(0.5) + 2(0.02)\sin(0.08\pi)]u_{44} + \lambda(u_{34} + u_{54})$

$u_{45} = [1 - 2(0.5) + 2(0.02)\sin(0.08\pi)]0.4857 + \lambda(0.8144 + 0) = 0.412$

Table 1

		x_0 0	x_1 0.2	x_2 0.4	x_3 0.6	x_4 0.8	x_5
t_0	0	0	4.0287	3.6655	0.1388	-1.6776	0
t_1	0.02	0	1.8328	2.0837	0.994	0.0694	0
t_2	0.04	0	1.0465	1.4186	1.079	0.4972	0
t_3	0.06	0	0.7145	1.0699	0.9633	0.5445	0
t_4	0.08	0	0.5403	0.8469	0.8144	0.4857	0
t_5	0.1	0	0.4288	0.6858	0.6744	0.412	0

Table 2

u_{1i}	$\Delta^{0+1}u_{1i}$	$\Delta^{0+2}u_{1i}$	$\Delta^{0+3}u_{1i}$	$\Delta^{0+4}u_{1i}$	$\Delta^{0+5}u_{1i}$
4.0287	-2.1959	1.4096	-0.9553	0.6588	-0.4574
1.8328	-0.7863	0.4543	-0.2965	0.2014	
1.0465	-0.332	0.1578	-0.0951		
0.7145	-0.1742	0.0627			
0.5403	-0.1115				
0.4288					

Table 3					
u_{2i}	$\Delta^{0+1}u_{2i}$	$\Delta^{0+2}u_{2i}$	$\Delta^{0+3}u_{2i}$	$\Delta^{0+4}u_{2i}$	$\Delta^{0+5}u_{2i}$
3.6655	-1.5818	0.9167	-0.6003	0.4096	-0.2827
2.0837	-0.6651	0.3164	-0.1907	0.1269	
1.4186	-0.3487	0.1257	-0.0638		
1.0699	-0.223	0.0619			
0.8469	-0.1611				
0.6858					

Table 4					
u_{3i}	$\Delta^{0+1}u_{3i}$	$\Delta^{0+2}u_{3i}$	$\Delta^{0+3}u_{3i}$	$\Delta^{0+4}u_{3i}$	$\Delta^{0+5}u_{3i}$
0.1388	0.8552	-0.7702	0.5695	-0.402	0.2766
0.994	0.085	-0.2007	0.1675	-0.1254	
1.079	-0.1157	-0.0332	0.0421		
0.9633	-0.1489	0.0089			
0.8144	-0.14				
0.6744					

Table 5					
u_{4i}	$\Delta^{0+1}u_{4i}$	$\Delta^{0+2}u_{4i}$	$\Delta^{0+3}u_{4i}$	$\Delta^{0+4}u_{4i}$	$\Delta^{0+5}u_{4i}$
-1.6776	1.747	-1.3192	0.9387	-0.6643	0.4811
0.0694	0.4278	-0.3805	0.2744	-0.1832	
0.4972	0.0473	-0.1061	0.0912		
0.5445	-0.0588	-0.0149			
0.4857	-0.0737				
0.412					

Table 6					
u_{i0}	$\Delta^{1+0}u_{i0}$	$\Delta^{2+0}u_{i0}$	$\Delta^{3+0}u_{i0}$	$\Delta^{4+0}u_{i0}$	$\Delta^{5+0}u_{i0}$
0	4.0287	-4.3919	1.2284	3.6454	-6.7355
4.0287	-0.3632	-3.1635	4.8738	-3.0901	
3.6655	-3.5267	1.7103	1.7837		
0.1388	-1.8164	3.494			
-1.6776	1.6776				
0					

Table 7

u_{i1}	$\Delta^{1+0}u_{i1}$	$\Delta^{2+0}u_{i1}$	$\Delta^{3+0}u_{i1}$	$\Delta^{4+0}u_{i1}$	$\Delta^{5+0}u_{i1}$
0	1.8328	-1.5819	0.2413	1.2644	-2.08
1.8328	0.2509	-1.3406	1.5057	-0.8156	
2.0837	-1.0897	0.1651	0.6901		
0.994	-0.9246	0.8552			
0.0694	-0.0694				
0					

Table 8

u_{i2}	$\Delta^{1+0}u_{i2}$	$\Delta^{2+0}u_{i2}$	$\Delta^{3+0}u_{i2}$	$\Delta^{4+0}u_{i2}$	$\Delta^{5+0}u_{i2}$
0	1.0465	-0.6744	-0.0373	0.5068	-0.6495
1.0465	0.3721	-0.7117	0.4695	-0.1427	
1.4186	-0.3396	-0.2422	0.3268		
1.079	-0.5818	0.0846			
0.4972	-0.4972				
0					

Table 9

u_{i3}	$\Delta^{1+0}u_{i3}$	$\Delta^{2+0}u_{i3}$	$\Delta^{3+0}u_{i3}$	$\Delta^{4+0}u_{i3}$	$\Delta^{5+0}u_{i3}$
0	0.7145	-0.3591	-0.1029	0.2527	-0.216
0.7145	0.3554	-0.462	0.1498	0.0367	
1.0699	-0.1066	-0.3122	0.1865		
0.9633	-0.4188	-0.1257			
0.5445	-0.5445				
0					

Table 10

u_{i4}	$\Delta^{1+0}u_{i4}$	$\Delta^{2+0}u_{i4}$	$\Delta^{3+0}u_{i4}$	$\Delta^{4+0}u_{i4}$	$\Delta^{5+0}u_{i4}$
0	0.5403	-0.2337	-0.1054	0.1483	-0.052
0.5403	0.3066	-0.3391	0.0429	0.0963	
0.8469	-0.0325	-0.2962	0.1392		
0.8144	-0.3287	-0.157			
0.4857	-0.4857				
0					

Table 11					
u_{i5}	$\Delta^{1+0}u_{i5}$	$\Delta^{2+0}u_{i5}$	$\Delta^{3+0}u_{i5}$	$\Delta^{4+0}u_{i5}$	$\Delta^{5+0}u_{i5}$
0	0.4288	-0.1718	-0.0966	0.114	-0.03
0.4288	0.257	-0.2684	0.0174	0.084	
0.6858	-0.0114	-0.251	0.1014		
0.6744	-0.2624	-0.1496			
0.412	-0.412				
0					

Since both the First and the Last Column of Table 1 contain 0, this means that

$\Delta^{0+1}u_{00} = \Delta^{0+2}u_{00} = \Delta^{0+3}u_{00} = \Delta^{0+4}u_{00} = \Delta^{0+5}u_{00} = 0$

And $\Delta^{0+1}u_{50} = \Delta^{0+2}u_{50} = \Delta^{0+3}u_{50} = \Delta^{0+4}u_{50} = \Delta^{0+5}u_{50} = 0$

From Table 2, we get

$\Delta^{0+1}u_{10} = -2.1959, \Delta^{0+2}u_{10} = 1.4096, \Delta^{0+3}u_{10} = -0.9553, \Delta^{0+4}u_{10} = 0.6588, \Delta^{0+5}u_{10} = -0.4574$

From Table 3

$\Delta^{0+1}u_{20} = -1.5818, \Delta^{0+2}u_{20} = 0.9167, \Delta^{0+3}u_{20} = -0.6003, \Delta^{0+4}u_{20} = 0.4096, \Delta^{0+5}u_{20} = -0.2827$

From Table 4

$\Delta^{0+1}u_{30} = 0.8552, \Delta^{0+2}u_{30} = -0.7702, \Delta^{0+3}u_{30} = 0.5695, \Delta^{0+4}u_{30} = -0.402, \Delta^{0+5}u_{30} = 0.2766$

From Table 5

$\Delta^{0+1}u_{40} = 1.747, \Delta^{0+2}u_{40} = -1.3192, \Delta^{0+3}u_{40} = 0.9387, \Delta^{0+4}u_{40} = -0.6643, \Delta^{0+5}u_{40} = 0.4811$

From Table 6

$\Delta^{1+0}u_{00} = 4.0287, \Delta^{2+0}u_{00} = -4.3919, \Delta^{3+0}u_{00} = 1.2284, \Delta^{4+0}u_{00} = 3.6454, \Delta^{5+0}u_{00} = -6.7355$

From Table 7

$\Delta^{1+0}u_{01} = 1.8328, \Delta^{2+0}u_{01} = -1.5819, \Delta^{3+0}u_{01} = 0.2413, \Delta^{4+0}u_{01} = 1.2644, \Delta^{5+0}u_{01} = -2.08$

From Table 8

$\Delta^{1+0}u_{02} = 1.0465, \Delta^{2+0}u_{02} = -0.6744, \Delta^{3+0}u_{02} = -0.0373, \Delta^{4+0}u_{02} = 0.5068, \Delta^{5+0}u_{02} = -0.6495$

From Table 9

$\Delta^{1+0}u_{03} = 0.7145, \Delta^{2+0}u_{03} = -0.3591, \Delta^{3+0}u_{03} = -0.1029, \Delta^{4+0}u_{03} = 0.2527, \Delta^{5+0}u_{03} = -0.216$

From Table 10

$\Delta^{1+0}u_{04} = 0.5403, \Delta^{2+0}u_{04} = -0.2337, \Delta^{3+0}u_{04} = -0.1054, \Delta^{4+0}u_{04} = 0.1483, \Delta^{5+0}u_{04} = -0.052$

From Table 11

$\Delta^{1+0}u_{05} = 0.4288, \Delta^{2+0}u_{05} = -0.1718, \Delta^{3+0}u_{05} = -0.0966, \Delta^{4+0}u_{05} = 0.114, \Delta^{5+0}u_{05} = -0.03$

The formula for determining the differences between two orders can be expressed generally as

$$\Delta^{m+n}u_{00} = \Delta^{m+0}u_{0n} - n\Delta^{m+0}u_{0n-1} + \frac{n(n-1)}{2!}\Delta^{m+0}u_{0n-2} - \cdots + (-1)^m\Delta^{m+0}u_{00}$$

$$\Delta^{n+m}u_{00} = \Delta^{0+n}u_{m0} - m\Delta^{0+n}u_{m-10} + \frac{m(m-1)}{2!}\Delta^{0+n}u_{m-20} - \cdots + (-1)^m\Delta^{0+n}u_{00}$$

$$\Delta^{1+1}u_{00} = \Delta^{1+0}u_{01} - \Delta^{1+0}u_{00} = 1.8328 - 4.0287 = -2.1959$$

$$\Delta^{1+2}u_{00} = \Delta^{1+0}u_{02} - 2\Delta^{1+0}u_{01} + \Delta^{1+0}u_{00} = 1.0465 - 2 \times 1.8328 + 4.0287 = 1.4096$$

$$\Delta^{2+1}u_{00} = \Delta^{2+0}u_{01} - \Delta^{2+0}u_{00} = -1.5819 - (-4.3919) = 2.8100$$

$$\Delta^{3+1}u_{00} = \Delta^{3+0}u_{01} - \Delta^{3+0}u_{00} = 0.2413 - 1.2284 = -0.9871$$

$$\Delta^{1+3}u_{00} = \Delta^{1+0}u_{03} - 3\Delta^{1+0}u_{02} + 3\Delta^{1+0}u_{01} - \Delta^{1+0}u_{00} = 0.7145 - 3 \times 1.0465 + 3 \times 1.8328 - 4.0287 = -0.9553$$

$$\Delta^{2+2}u_{00} = \Delta^{2+0}u_{02} - 2\Delta^{2+0}u_{01} + \Delta^{2+0}u_{00} = -0.6744 - 2 \times -1.5819 - 4.3919 = -1.9025$$

$$\Delta^{1+4}u_{00} = \Delta^{1+0}u_{04} - 4\Delta^{1+0}u_{03} + 6\Delta^{1+0}u_{02} - 4\Delta^{1+0}u_{01} + \Delta^{1+0}u_{00} = 0.5403 - 4 \times 0.7145 + 6 \times 1.0465 - 4 \times 1.8328 + 4.0287 = 0.6588$$

$$\Delta^{4+1}u_{00} = \Delta^{4+0}u_{01} - \Delta^{4+0}u_{00} = 1.2644 - 3.6454 = -2.3810$$

$$\Delta^{3+2}u_{00} = \Delta^{3+0}u_{02} - 2\Delta^{3+0}u_{01} + \Delta^{3+0}u_{00} = -0.0373 - 2 \times 0.2413 + 1.2284 = 0.7085$$

$$\Delta^{2+3}u_{00} = \Delta^{2+0}u_{03} - 3\Delta^{2+0}u_{02} + 3\Delta^{2+0}u_{01} - \Delta^{2+0}u_{00} = -0.3591 - 3 \times (-0.6744) + 3 \times -1.5819 - (-4.3919) = 1.3103$$

Interpolating polynomials in two variables up to the difference of the fifth degree requires the following formula:

$$u(x,t) = u_{00} + \left[\frac{(x-x_0)}{h}\Delta^{1+0}u_{00} + \frac{(t-t_0)}{k}\Delta^{0+1}u_{00}\right]$$
$$+\frac{1}{2!}\left[\frac{(x-x_0)(x-x_1)}{h^2}\Delta^{2+0}u_{00} + \frac{2(x-x_0)(t-t_0)}{hk}\Delta^{1+1}u_{00} + \frac{(t-t_0)(t-t_1)}{k^2}\Delta^{0+2}u_{00}\right]$$
$$+\frac{1}{3!}\left[\frac{(x-x_0)(x-x_1)(x-x_2)}{h^3}\Delta^{3+0}u_{00} + \frac{3(x-x_0)(x-x_1)(t-t_0)}{h^2k}\Delta^{2+1}u_{00} + \frac{3(x-x_0)(t-t_0)(t-t_1)}{hk^2}\Delta^{1+2}u_{00} + \frac{(t-t_0)(t-t_1)(t-t_2)}{k^3}\Delta^{0+3}u_{00}\right]$$
$$+\frac{1}{4!}\left[\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{h^4}\Delta^{4+0}u_{00} + \frac{4(x-x_0)(x-x_1)(x-x_2)(t-t_0)}{h^3k}\Delta^{3+1}u_{00} + \frac{6(x-x_0)(x-x_1)(t-t_0)(t-t_1)}{h^2k^2}\Delta^{2+2}u_{00} + \frac{4(x-x_0)(t-t_0)(t-t_1)(t-t_2)}{hk^3}\Delta^{1+3}u_{00} + \frac{(t-t_0)(t-t_1)(t-t_2)(t-t_3)}{k^4}\Delta^{0+4}u_{00}\right]$$

$$+\frac{1}{5!}\left[\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{h^5}\Delta^{5+0}u_{00}+\frac{5(x-x_0)(x-x_1)(x-x_2)(x-x_3)(t-t_0)}{h^4k}\Delta^{4+1}u_{00}+\frac{10(x-x_0)(x-x_1)((x-x_2)(t-t_0)(t-t_1)}{h^3k^2}\Delta^{3+2}u_{00}+\frac{10(x-x_0)(x-x_1)(t-t_0)(t-t_1)(t-t_2)}{h^2k^3}\Delta^{2+3}u_{00}+\frac{5(x-x_0)(t-t_0)(t-t_1)(t-t_2)(t-t_3)}{hk^4}\Delta^{1+4}u_{00}+\frac{(t-t_0)(t-t_1)(t-t_2)(t-t_3)(t-t_4)}{k^5}\Delta^{0+5}u_{00}\right] \tag{17}$$

$$u(x,t)=$$

$$\left[\frac{(x-0)}{0.2}(4.0287)+\frac{(t-0)}{0.02}(0)\right]$$

$$+\frac{1}{2!}\left[\frac{(x-0)(x-0.2)}{(0.2)^2}(-4.3919)+\frac{2(x-0)(t-0)}{(0.2)(0.02)}(-2.1959)+\frac{(t-0)(t-0.02)}{(00.2)^2}(0)\right]$$

$$+\frac{1}{3!}\left[\frac{(x-0)(x-0.2)(x-0.4)}{(0.2)^3}(1.2284)+\frac{3(x-0)(x-0.2)(t-0)}{(0.2)^2(0.02)}(2.8100\ )+\frac{3(x-0)(t-0)(t-0.02)}{(0.2)(0.02)^2}(1.4096\ )+\frac{(t-0)(t-0.02)(t-0.04)}{(0.02)^3}(0)\right]$$

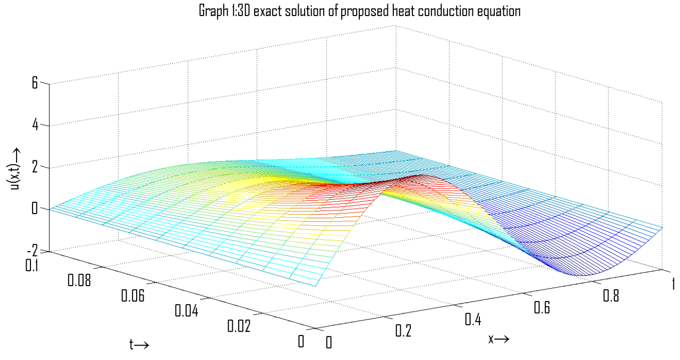
$$+\frac{1}{4!}\left[\frac{(x-0)(x-0.2)(x-0.4)(x-0.4)}{(0.2)^4}(3.6454)+\frac{4(x-0)(x-0.2)(x-0.4)(t-0)}{(0.2)^3(0.02)}(-0.9871)+\frac{6(x-0)(x-0.2)(t-0)(t-0.02)}{(0.2)^2(0.02)^2}(-1.9025)+\frac{4(x-0)(t-0)(t-0.02)(t-0.04)}{(0.2)(0.02)^3}(-0.9553\ )+\frac{(t-0)(t-0.02)(t-0.04)(t-0.06)}{(0.02)^4}(0)\right]$$

$$+\frac{1}{5!}\left[\frac{(x-0)(x-0.2)(x-0.4)(x-0.6)(x-0.8)}{(0.2)^5}(-6.7355\ )+\frac{5(x-0)(x-0.2)(x-0.4)(x-0.6)(t-0)}{(0.2)^4(0.02)}(-2.3810)+\frac{10(x-0)(x-0.2)((x-0.4)(t-0)(t-0.02)}{(0.2)^3(0.02)^2}(0.7085)+\frac{10(x-0)(x-0.2)(t-0)(t-0.02)(t-0.04)}{(0.2)^2(0.02)^3}(1.3103)+\frac{5(x-0)(t-0)(t-0.02)(t-0.04)(t-0.06)}{(0.2)(0.02)^4}(0.6588)+\frac{(t-0)(t-0.02)(t-0.04)(t-0.06)(t-0.08)}{(0.02)^5}(0)\right] \tag{18}$$

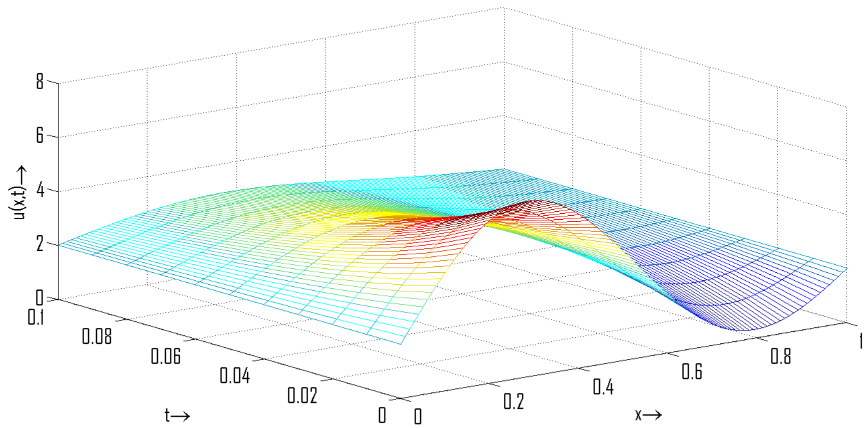
$$u(x,t)=20.1435x-54.8987x(x-0.2)-\ 0.2196xt+\ 25.5917x(x-0.2)(x-0.4)+0.0011xt(x-0.2)+0.000056384xt(t-0.02)+\ 94.9323x(x-0.2)(x-0.4)(x-0.6)-0.000026323xt(x-0.2)(x-0.4)-0.00000761xt(x-0.2)(t-0.02)-0.00000025475xt(t-0.02)(t-0.04)-0.000017961x(x-0.2)(x-0.4)(x-0.6)(x-0.8)-0.0000031747xt(x-0.2)(x-0.4)(x-0.6)+0.00000018893xt(x-0.2)((x-0.4)(t-0.02)+0.000000034941xt(x-0.2)(t-0.02)(t-0.04)+0.0000000008784xt(t-0.02)(t-0.04)(t-0.06) \tag{19}$$

4. RESULTS AND DISCUSSION

Graph 1:3D exact solution of proposed heat conduction equation



Graph 2:3D approximate solution of proposed heat conduction equation by DIM



Graph 3:2D approximate solution of proposed heat conduction equation by DIM for different values of temporal coordinate

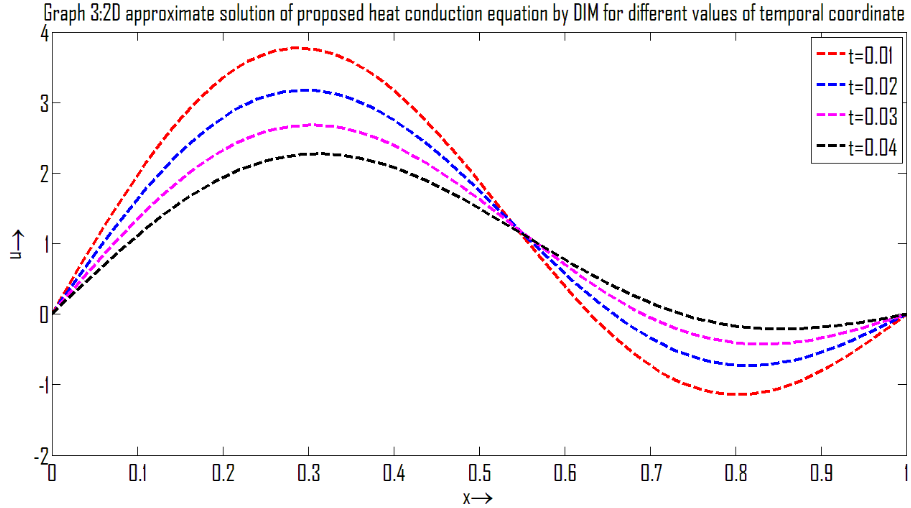


Table 12: Estimation of the error between the exact and DIM solution of the proposed heat conduction equation												
x	$t = 0.01$			$t = 0.02$			$t = 0.03$			$t = 0.04$		
	Exact	DIM	Error	Exact	DIM	Error	Exact	DIM	Error	Exact	DIM	Error
0.1	1.9678	1.9307	0.0371	1.6267	1.5961	0.0306	1.3458	1.3206	0.0252	1.1147	1.094	0.0207
0.3	3.7647	3.7008	0.0639	3.1743	3.1209	0.0534	2.6808	2.6361	0.0447	2.2678	2.2304	0.0374
0.5	1.8792	1.8669	0.0123	1.7547	1.7422	0.0125	1.6284	1.6157	0.0127	1.5019	1.4893	0.0126
0.7	-0.7242	-0.6802	0.044	-0.3351	-0.302	0.0331	-0.0459	-0.0217	0.0242	0.1623	0.1793	0.017
0.9	-0.8065	-0.7769	0.0296	-0.5422	-0.5194	0.0228	-0.3394	-0.322	0.0174	-0.1865	-0.1736	0.0129

5. CONCLUDING REMARKS

In conclusion, this paper has introduced a novel double interpolation approach for approximating the heat conduction equation with source/sink terms. We have demonstrated the effectiveness of our method in accurately capturing the spatial and temporal variations of the temperature field, thereby addressing the challenges posed by complex heat conduction problems. By leveraging interpolation techniques both spatially and temporally, our approach offers several advantages over traditional numerical methods. The double interpolation method provides a more refined estimation of the temperature distribution, leading to improved accuracy in modeling heat transfer processes with source or sink terms. Moreover, our approach is computationally efficient, making it suitable for practical applications in various fields such as thermal engineering, materials science, and environmental modeling. Through numerical experiments and comparisons with existing methods, we have shown that the double interpolation approach outperforms alternative techniques in terms of accuracy and computational efficiency. Our method offers a promising avenue for further research and development in the field of numerical heat transfer analysis.

In future work, it would be valuable to explore extensions and refinements of the double interpolation approach, as well as its application to more complex heat conduction problems. Additionally, investigations into the optimization and implementation of the method on parallel computing architectures could further enhance its scalability and applicability to large-scale simulations. Overall, the double interpolation approach presented in this paper represents a significant advancement in the numerical approximation of heat conduction equations with source/sink terms, with implications for improving our understanding and modeling of thermal processes in diverse real-world systems.

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