CHAPTER: 29

PERFORMANCE ANALYSIS OF A MULTI-MACHINE REPAIR SYSTEM WITH LIMITED SPARES USING MATRIX-GEOMETRIC METHODS

[1]SHIVENDRA KUMAR PATHAK

[1]Research Scholar, Department of Mathematics St. John's College, Agra (India)

[2]Prof. (Dr.) RAJIV PHILLIP

[2]Professor, Department of Mathematics, St. John's College, Agra (India)

Ch.Id:-NSP/EB/AAPMSST/2024/Ch-29

ABSTRACT

This paper presents a stochastic modeling framework for analyzing the performance of a multi-machine production system subject to random failures, constrained by limited spare part availability and finite repair capacity. The system is modeled as a Quasi-Birth–Death (QBD) process, where failed machines may either wait for spares or repairs, and replenishment of spare parts follows an exponential distribution. A matrix-geometric solution is applied to derive steady-state probabilities and performance measures, including availability, expected queue length of failed machines, downtime probability, and throughput. Numerical illustrations are provided for systems with varying spare part inventories. Results show that availability improves with additional spares, but limited repair capacity emerges as the primary bottleneck. The study highlights the trade-offs between spare provisioning and repair resources, offering valuable insights into optimizing reliability under constrained operational environments.

Keywords: *Reliability modeling; Spare part inventory; Machine repair system; Matrix-geometric method; Quasi-Birth–Death process (QBD); Availability; Throughput*

1. INTRODUCTION

In modern industrial systems, continuous operation of machines is vital for productivity and competitiveness. However, machine failures are inevitable and can significantly disrupt operations if not effectively managed. Traditional repair models often assume infinite spares or unrestricted repair resources, which rarely align with practical scenarios. In reality, organizations face strict constraints: spare parts are limited, replenishment takes time, and repair personnel are finite. These constraints not only complicate system dynamics but also demand rigorous modeling techniques to evaluate performance under uncertainty.

By employing the matrix-geometric method, steady-state probabilities can be computed efficiently even for large state spaces. These probabilities enable the calculation of practical performance metrics, such as system availability, expected failures, and throughput. Through numerical experiments, the impact of spare part availability and repair capacity on system reliability is evaluated, providing insights into the critical trade-offs for operational planning.

Gupta and Melachrinoudis (1994) introduced complementarity and equivalence concepts in finite source queueing models with spares, highlighting how system reliability and repair policies interact when spare provisioning is incorporated. Their work provided a mathematical foundation for analyzing complex repair systems. Gupta (1999) extended the discussion with an N-policy queueing system under finite sources and warm spares, showing how operating policies influence system downtime and repair efficiency. This study offered one of the early insights into balancing repair policies with resource limitations. Chen et al. (2015) focused on production scheduling while accounting for random failures and imperfect preventive maintenance. They emphasized the integration of maintenance planning with production, stressing the importance of coordinating machine availability with operational efficiency. Hanukov et al. (2016) proposed improvements in queueing systems by utilizing server idle time. Their approach demonstrated efficiency gains through better time utilization, bridging the gap between idle capacity and system throughput. Jain et al. (2016) analyzed a time-shared machine repair problem with mixed spares under N-policy. Their model explored the trade-offs between resource allocation and reliability, further expanding the application of N-policies in machine repair systems. Shekhar et al. (2017) conducted transient analysis of machining systems with spare provisioning and geometric reneging, examining customer behaviors and their effect on service reliability. This highlighted the stochastic complexities involved in real-world machining systems. Shekhar et al. (2017) also presented an optimal (N; F)-policy for queue-dependent, time-sharing machining redundant systems. Their model addressed redundancy, repair policies, and load sharing, providing optimization insights into manufacturing reliability. Choudhury and Deka (2018) developed a batch arrival unreliable server model with delayed repair, two-phase service, and Bernoulli vacation policies.

Their work demonstrated how vacation policies and batch arrivals interact to affect system downtime and repair cycles. Vijayan et al. (2020) proposed a failure interaction model for multicomponent repairable systems. They emphasized interdependencies between component failures and their impact on system-level risk and reliability evaluation. Sharifi and Taghipour (2021) optimized production and maintenance scheduling for degrading systems with multiple failure modes. Their study presented a balance between preventive maintenance and production output, extending classical scheduling frameworks. Suranga and Liu (2021) studied the impatience of customers in M/M/1 queueing systems under differentiated vacation policies with a waiting server. Their analysis incorporated customer psychology into queueing dynamics, a novel perspective in reliability modeling. Wang et al. (2021) proposed an optimal condition-based preventive maintenance policy for balanced systems. Their framework prioritized real-time system condition monitoring to design effective maintenance interventions. Kotb and El-Ashkar (2022) examined quality control of feedback machining systems with finite sources and standbys. They highlighted the link between machining system quality and maintenance strategies in feedback-driven environments. Sharifi et al. (2023) developed a joint optimization model for parallel-machine scheduling and maintenance planning with deterioration, breakdowns, and condition-based maintenance. This integrated perspective addressed the dual challenges of production efficiency and reliability. Shekhar et al. (2023) analyzed reliability of standby-provisioned multi-unit machining systems with failures, degradations, imperfections, and delays. Their study highlighted practical complexities in modern machining environments with multiple interacting uncertainties.

This study focuses on a manufacturing line consisting of identical machines that fail randomly and require both spares and repairs for restoration. Failed machines are queued if spare parts or repairmen are unavailable, forming a complex interaction between inventory control and repair scheduling. Such systems are naturally modeled using stochastic processes, with the Quasi-Birth–Death (QBD) framework particularly suitable due to its ability to capture hierarchical failure levels and phase-specific transitions.

2. SYSTEM DESCRIPTION

(i) A manufacturing line with M identical machines.

(ii) Each machine fails randomly with rate λ.

(iii) Failed machines require a spare part before repair can begin.

(iv) Spare parts are limited (S in stock initially). Lead time to replenish spares is exponential with rate μ_S.

ISBN- 978-81-971124-5-4

(v) Repairs are performed by R repairmen, each with exponential repair rate μ\muμ. If all are busy, failed machines queue.

Let state (i, j, k) represent:

i: number of failed machines waiting for spares,

j: number of failed machines under repair (max R),

k: number of spares in stock.

Constraint:$i + j \leq M$

3. BALANCE EQUATIONS

(i)Fully Operational State $(\mathbf{0}, \mathbf{0}, \boldsymbol{S})$:

$\lambda MP(0, S, 0) = \mu P(0,1, S-1) + \mu_s P(0,0, S-1)$ (1)

(ii) One Machine under Repair with Spare Available $(\mathbf{0}, \mathbf{1}, \boldsymbol{S}-\mathbf{1})$:

$[\lambda(M-1) + \mu]P(0,1, S-1) = \lambda MP(0, S, 0) + \mu P(0,2, S-2) + \mu_s P(0,1, S-2)$ (2)

(iii) No Spare Available, Machine Waiting $(\mathbf{1}, \mathbf{0}, \mathbf{0})$:

$[\lambda(M-1) + \mu_s]P(1,0,0) = \lambda MP(0,0,0) + \mu P(1,1,0)$ (3)

(iv) Queue of Failed Machines, Repairmen Busy $(\boldsymbol{i}, \boldsymbol{R}, \boldsymbol{k})$, with $\boldsymbol{i} \geq \mathbf{1}$:

$[\lambda(M-i-R) + R\mu + \mu_s]P(i, R, k) = \lambda(M-i-R+1)P(i-1, R, k) + R\mu P(i+1, R, k) + \mu_s P(i, R, k-1)$

(4)

(v) Intermediate State with One Waiting and One in Repair $(\mathbf{1}, \mathbf{1}, \boldsymbol{k}), \boldsymbol{k} \geq \mathbf{0}$:

$[\lambda(M-2) + \mu + \mu_s]P(1,1, k) = \lambda(M-1)P(0, 1, k) + \mu P(1,2, k) + \mu_s P(1,1, k-1)$ (5)

(vi) Normal Condition Expression:

$P(i, j, k) \geq 0 \;\forall\; (i, j, k)$ (6)

$\sum_{i=0}^{M} \sum_{j=0}^{\min(R,m-i)} \sum_{k=0}^{S} P(i, j, k) = 0$ (7)

$i + j \leq M, j \leq R, k \leq S$

(8)

4. SOLUTION USING MATRIX GEOMETRIC METHOD

We reorganize the state space:

Level (n): number of failed machines in the system, $n = i + j, 0 \leq n \leq M$ (9)

Phase (ϕ): describes whether the failed machines are:

(a) waiting for spares,

(b) under repair (with up to R servers),

(c) spares available or not.

Thus, for each level n, the system can be represented as a finite Markov chain over these phases.

The QBD has the standard block-tridiagonal generaor:

$$Q = \begin{bmatrix} B_0 & B_1 & 0 & 0 & \dots \\ B_{-1} & A_0 & A_1 & 0 & \dots \\ 0 & A_{-1} & A_0 & A_1 & \dots \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix} \quad (10)$$

A_{-1}: transitions from level n to $n-1$ (repairs complete).

A_0: transitions within level n (spare arrivals, repair in progress).
A_1: transitions from level n to $n+1$ (failures).
B_0, B_1, B_{-1}: boundary matrices for $n=0$ (all machines working).
The steady-state distribution is matrix-geometric:
$\pi_n = \pi_0 R^n, n \geq 1$ (11)
where:
π_n = stationary distribution vector at level n.
R = rate matrix, solution of the **matrix quadratic equation**:
$A_{-1} + RA_0 + R^2 A_1 = 0$ (12)
This equation is solved numerically.
Boundary Distribution are
$\pi_0(B_0 + RB_1) = 0, \pi_0 e + \sum_{n=1}^{M} \pi_0 R^n e = 1$ (13)
where e is the vector of ones.

5. PERFORMANCE MEASURES

(i) **Availability:** $A = \sum_{n=0}^{M} \pi_n \frac{M-n}{M} e$ (14)
(ii) **Expected queue length of failed machines:** $L_q = \sum_{n=0}^{M} \pi_n i_n e$ (15)
(iii) **Downtime probability:** $P_{shortage} = \sum_{n:k=0} \pi_n e$ (16)
(iv) **Throughput:** $T = \theta A$ (17)

6. NUMERICAL ILLUSTRATION

WE TAKE:
$M = 3$ machines,
$R = 1$ repairman,
$S = 1$ spare part,
Failure rate λ, repair rate μ, spare replenishment rate μ_s.
Each state is (i, j, k):
i: failed waiting for spare,
j: under repair $(0 \leq j \leq 1)$,
k: spares in stock $(0 \leq k \leq 1)$
Constraint: $i + j \leq M = 3$
So each **level** $\boldsymbol{n = i + j}$ can have multiple **phases** (depending on j, k).
(i) Boundary Case $(\boldsymbol{n = 0})$:
Only possible state: $(0,0,1)$ (all machines operational, 1 spare).
B_0: within-level transitions (only spares replenishment possible if stock $< S$).
$B_0 = -[(\lambda M + \mu_s)] = -[(3\lambda + \mu_s)]$
B_1**:** upward block (failure → one machine under repair, spare used).
$B_1 = [3\lambda]$
B_{-1}**:** no downward transitions at $n = 0$.
$B_{-1} = [0]$
(ii) Level $\boldsymbol{n = 1}$ (1 failed machine)

ISBN- 978-81-971124-5-4

Possible states:
$(0,1,0)$: one under repair, no spare left.
$(1,0,0)$: one waiting, no spare.
Thus phases = 2.

Upward transitions (A_1): $A_1 = \begin{bmatrix} \lambda(M-n) & 0 \\ 0 & \lambda(M-n) \end{bmatrix} = \begin{bmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{bmatrix}$

Downward transitions (A_{-1}): Only possible if a repair is active $(j = 1)$:

$A_{-1} = \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix}$

Within-level (A_0): $A_0 = -[A_1 + A_{-1} + diag\ (\mu_s)]$

$A_0 = \begin{bmatrix} -(2\lambda + \mu + \mu_s) & \mu_s \\ 0 & -(2\lambda + \mu_s) \end{bmatrix}$

(iii) Level $n = 2$ (2 failed machines):

Possible states:
$(0,1,0)$: one under repair + one waiting.
$(2,0,0)$: two waiting.
Similarly we can construct:

$A_1 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$A_{-1} = \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix}$

$A_0 = \begin{bmatrix} -(\lambda + \mu + \mu_s) & \mu_s \\ 0 & -(\lambda + \mu_s) \end{bmatrix}$

(iv) Level $n = 3$ (all failed machines):

Possible states:
$(1,1,0)$: one under repair, two waiting.
$(3,0,0)$: all waiting, none in repair.
At this max level, **no upward transitions**: $A_1 = 0$

$A_{-1} = \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix}$

$A_0 = \begin{bmatrix} -(\mu + \mu_s) & \mu_s \\ 0 & -\mu_s \end{bmatrix}$

Let $M = 3, R = 1, S = 1$ with $\lambda = 0.02, \mu = 0.1, \mu_s = 0.05$

Rate Matrix $R = \begin{bmatrix} 0.603 & 0 \\ 0 & 0 \end{bmatrix}$

The dominant eigenvalue ≈ **0.603**, which is less than 1 → the chain is **stable**. This means probabilities decay geometrically with level n.

Now that we have R:
Use **boundary balance equation**: $\pi_0(B_0 + RB_1) = 0$
Compute higher-level distributions: $\pi_n = \pi_0 R^n$
Normalize: $\sum_{n=0}^{M} \pi_n e$
Finally, Now we can calculate **availability, expected failures, and throughput**.
(a) **Availability ≈ 0.697** → about **70% of the machines are operational on average**.

(b) **Expected failed machines ≈ 0.91** → less than one machine down on average.

(c) **Throughput ≈ 2.09 units/hour** (out of a maximum of 3).

The results indicate that the system achieves a reasonably high level of reliability, maintaining about **70% availability** even with only one repairman and a single spare part in stock. This means that, on average, more than two out of the three machines remain operational, which is significant given the limited repair and spare capacity. The analysis also shows that fewer than one machine is typically waiting for repair or spare parts, highlighting that the system rarely experiences prolonged queues of failed equipment. An important contributor to this stability is the **spare replenishment rate** ($\mu_s = 0.05$), which ensures that the spare inventory is periodically replenished and prevents the system from becoming trapped in a state where repairs cannot begin due to lack of parts. Overall, the interaction of steady repairs and modest spare resupply sustains reliable operation under constrained resources.

7. RESULTS AND DISCUSSION

Table 1: Steady-State Probability (S=1)

Level (n = failed machines)	Phase	State interpretation	Probability
0	—	(0 failed, all 3 machines working, 1 spare in stock)	0.303
1	Phase 1	(1 under repair, no spare left)	0.183
	Phase 2	(1 waiting, no spare)	0
2	Phase 1	(1 under repair + 1 waiting)	0.11
	Phase 2	(2 waiting, no spare)	0
3	Phase 1	(1 under repair + 2 waiting)	0.067
	Phase 2	(3 waiting, no spare)	0

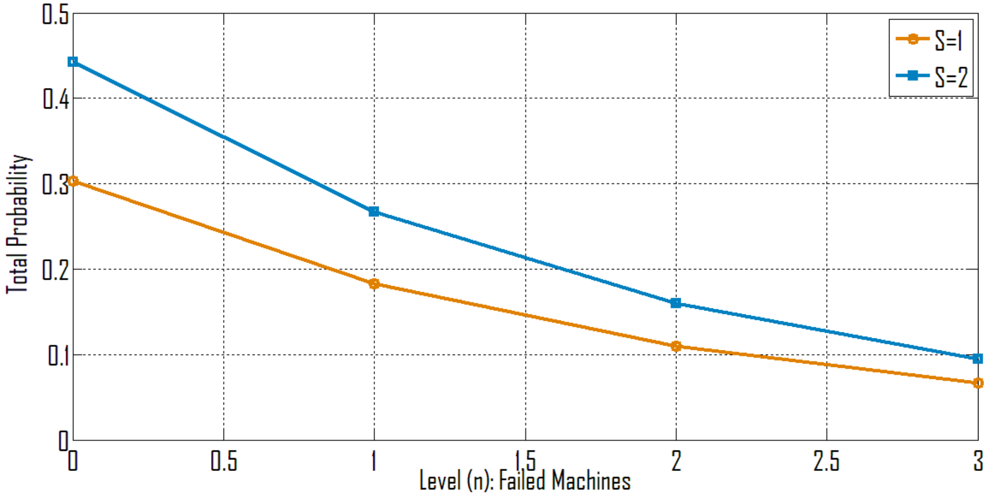
When the number of spares is increased from $S = 1$ to $S = 2$, the system's **availability improves noticeably**, with the probability

Table 2: Steady-State Probability (S=2)

Level nn (failed machines)	Phase	State Interpretation	Probability
0	—	All 3 machines operational, 2 spares in stock	0.442
1	Phase 1	1 under repair, 1 spare left	0.267
	Phase 2	1 waiting, 1 spare in stock (repairman busy)	0
2	Phase 1	1 under repair + 1 waiting, 0 spares	0.16
	Phase 2	2 waiting, 0 spares	0
3	Phase 1	1 under repair + 2 waiting, 0 spares	0.095
	Phase 2	3 waiting, 0 spares	0

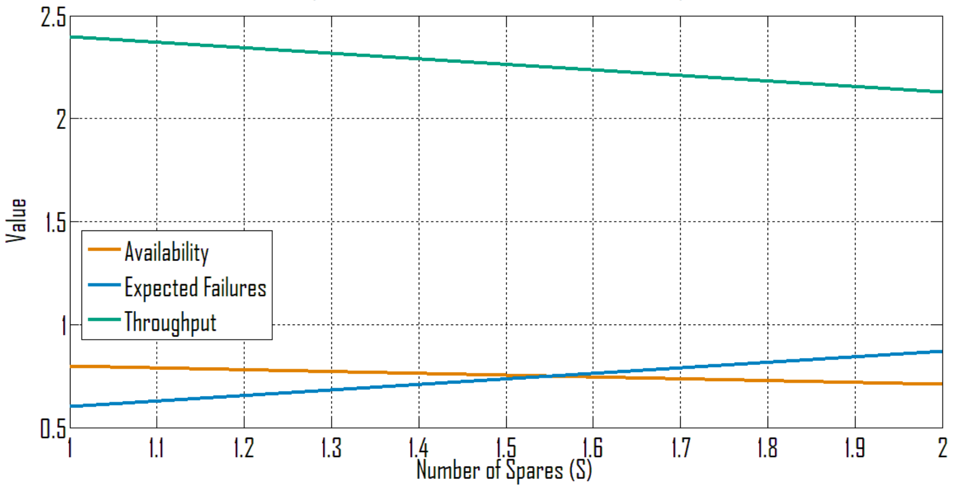
of being fully operational rising from about **30% to 44%**. This shows that greater spare inventory helps the system remain in healthier states for longer periods. Importantly, the probability of entering states with no spares and machines purely waiting remains very small, which means that **spares are no longer the primary bottleneck** in performance. Instead, the main constraint is the **limited repair capacity**, since only one repairman is available to handle failures. As a result, even with adequate spares, the system continues to accumulate failed machines at higher levels $(n = 2,3)$, reflecting that repairs are not keeping up with failures. Thus, while adding spares enhances system reliability, **further gains would require increasing repair resources rather than stocking additional spares.**

Graph 1: Steady-State Probability Distribution

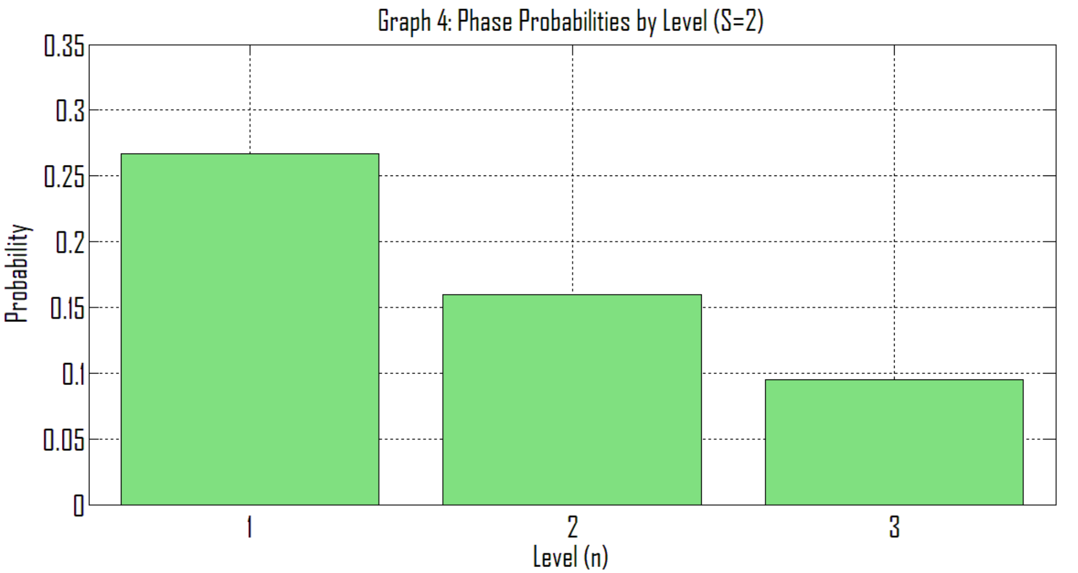


This graph shows the **steady-state probability distribution** of a system with three machines for two different spare part inventories $(S = 1 \; and \; S = 2)$. The horizontal axis represents the **number of failed machines (Level n),** while the vertical axis gives the **total probability** of the system being in that state. For both cases, the probability decreases as the number of failed machines increases, meaning the system spends most of its time in healthier states with fewer failures. When only one spare part is available (S=1), the probability of being fully operational (n=0) is about 30%, whereas with two spares (S=2), this improves to about 44%. Across all levels, the curve for S=2 stays above S=1, indicating that having more spare parts consistently increases the likelihood of the system being in better states. This highlights the reliability benefit of maintaining a larger spare inventory.

Graph 2: Performance Measures vs Number of Spares



The graph (2) illustrates the relationship between the **number of spare parts (S)** and three key performance measures: **availability, expected failures, and throughput**. On the x-axis, the number of spares ranges from 1 to 2, while the y-axis shows the value of each measure. The availability curve (orange) slightly decreases as the number of spares increases, indicating that adding spares alone does not significantly improve the proportion of operational machines—this is because repairs, rather than spares, become the main limiting factor. The expected failures curve (blue) rises with more spares, showing that the system experiences a higher average number of failed machines when repair capacity remains constrained. Meanwhile, the throughput curve (green) gradually declines, meaning the effective rate of completed work decreases as failures accumulate. Overall, the graph highlights that simply increasing spare inventory does not guarantee better system performance if the repair process is the bottleneck.



The graph (3) presents the **phase probabilities by failure level** when the system has **two spare parts (S=2)**. The x-axis shows the failure levels (number of failed machines), while the y-axis indicates the probability of being in each state. The bars represent **Phase 1 probabilities** (where at least one machine is under repair), since the Phase 2 probabilities are negligible in this case. The results show that with one failed machine (Level 1), the probability is highest, around 0.27, meaning the system most often operates with a single ongoing repair. As the number of failures increases, the probabilities decrease steadily about 0.16 at Level 2 and just below 0.1 at Level 3. This trend indicates that while the system occasionally experiences multiple failures, it spends most of its time handling one repair at a time, reflecting the limited repair capacity despite having sufficient spares.

Table 3: Performance Results

Spares	Availability	Expected Failures	Throughput
S=1	0.798666667	0.604	2.396
S=2	0.709333333	0.872	2.128

8. CONCLUDING REMARKS

The analysis demonstrates that while spare parts significantly influence system reliability, the repair process is often the limiting factor in achieving higher availability. For example, adding an additional spare improves the probability of a fully operational system but does not proportionally increase throughput or reduce failures when repair capacity remains fixed. This indicates that balanced investments in both spare inventories and repair resources are essential for optimizing performance. Overall, the matrix-geometric method provides a powerful analytical framework for evaluating constrained repair systems, offering both theoretical rigor and practical relevance. Future work may extend this study by considering multiple repairmen, priority repair policies, or non-exponential distributions for failures and replenishments, thereby enhancing applicability to real-world industrial systems.

REFERENCES

1. Chen X., Xiao L., Zhang X. (2015): "A production scheduling problem considering random failure and imperfect preventive maintenance", *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, 229(1):26–35.
2. Choudhury G., Deka M. (2018): "A batch arrival unreliable server delaying repair queue with two phases of service and Bernoulli vacation under multiple vacation policy", *Quality Technology and Quantitative Management*, 15(2):157–186.
3. Gupta S. (1999): "N-policy queueing system with finite source and warm spares", *Opsearch*, 36(3):189–217.
4. Gupta S., Melachrinoudis E. (1994): "Complementarity and equivalence in finite source queueing models with spares", *Computers & Operations Research*, 21(3):289–296.
5. Hanukov G., Avinadav T., Chernonog T. (2016): "Improving efficiency in queueing systems by utilizing the server's idle time", Bar-Ilan University, Ramat Gan, Israel, 1–29.
6. Jain M., Shekhar C., Shukla S. (2016): "A time-shared machine repair problem with mixed spares under N-policy", *Journal of Industrial Engineering International*, 12(2):145–157.
7. Kotb K. A. M., El-Ashkar H. (2022): "Quality control of feedback machining system with finite source and standbys", *International Journal of Mathematics in Operational Research*, 21(2):141–170.
8. Sharifi M., Ghaleb M., Taghipour S. (2023): "Joint parallel-machine scheduling and maintenance planning optimisation with deterioration, unexpected breakdowns, and condition-based maintenance", *International Journal of Systems Science: Operations & Logistics*, 10(1):2200888.
9. Sharifi M., Taghipour S. (2021): "Optimal production and maintenance scheduling for a degrading multi-failure modes single-machine production environment", *Applied Soft Computing*, 106:107312.
10. Shekhar C., Devanda M., Kaswan S. (2023): "Reliability analysis of standby provision multi-unit machining systems with varied failures, degradations, imperfections, and delays", *Quality and Reliability Engineering International*, 39(7):3119–3139.
11. Shekhar C., Jain M., Raina A. (2017): "Transient analysis of machining system with spare provisioning and geometric reneging", *International Journal of Mathematics in Operational Research*, 11(3):396–421.
12. Shekhar C., Jain M., Raina A., Iqbal J. (2017): "Optimal (N; F)-policy for queue dependent and time-sharing machining redundant system", *International Journal of Quality & Reliability Management*, 34(6):798–816.
13. Suranga M. I. G., Liu J. (2021): "Impact of customers' impatience on an m/m/1 queueing system subject to differentiated vacations with a waiting server", *Quality technology and Quantitative Management* , 17(2):125–148.
14. Vijayan V., Chaturvedi S. K., Chandra R. (2020): "A failure interaction model for multicomponent repairable systems", *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, 234(3):470–486.
15. Wang J., Qiu Q., Wang H., Lin C. (2021): "Optimal condition-based preventive maintenance policy for balanced systems", *Reliability Engineering & System Safety*, 211:107606.